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THE ANALYTICAL GEOMETRY

OF

THE STRAIGHT LINE AND THE CIRCLE

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BY

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PREFACE

This book is intended to be an introduction to the formal study of analytical geometry. The advance from ordinary geometry to analytical geometry is often beset with difficulties for the beginner, who usually finds it hard to think and visualise in terms of the new methods. With a view to making his road as easy as possible I have dealt only with the simpler forms of the equations belonging to the straight line and circle, in the first six chapters of this book, reserving the harder forms for the remaining chapters. To the same end I have employed a method for establishing the linear locus which involves geometry of a very simple and familiar kind, and have afterwards applied it when finding the distance of a point from a line, a formula the working-out of which is frequently a stumbling-block to the learner. In this way it is hoped that the progress of the student will be along a path of gentle slope, from well-known geometrical methods to the highly analytical processes involved in *Joachimsthal's Ratio Equation*. In many cases I have worked out introductory examples of a numerical type before dealing with a piece of general theory, in order that the reader's mind may have a clear picture to help him to follow the general argument. Considerable attention has been paid to the text and diagrams dealing with pencils of lines and circles, as a firm grasp of these matters is essential to the student who intends to proceed to the higher mathematics. Bearing in mind the needs of the private student I have been very full in all the discussions and have given many illustrative examples.

To provide a collection of suitable exercises has been a work of no small labour. The questions hitherto set in examination papers have rarely involved a geometrical figure, but have been mere pieces of algebraic work. This source of examples, usually so fruitful, has therefore been non-existent for me, as my aim has been to show the student that analytical geometry does really deal with geometry, and is not all algebra, a point on which he is often dubious at the end of the usual preliminary course. Hence while setting a sufficient number of drill examples, I have adapted many familiar deductions, or have myself evolved a suitable geometrical configuration. Examples worked out in the text have often been inserted among the examples at the ends of the chapters, so that the student may turn back and compare his own methods with those of the book, noting where he may improve when his own are inferior, and deriving encouragement when they are superior. Whenever possible, he should solve his problem by ordinary geometry after he has worked it out by analysis. In no case should he neglect to carry out the graphical part of those questions in which he has been instructed to use squared paper.

I wish to acknowledge gratefully my very deep indebtedness to Dr. W. P. Milne, Clifton College, for invaluable suggestions and criticism. I have also to thank Dr. Charles M'Leod, Senior Mathematical Master, Grammar School, Aberdeen, for advice on some points; Mr. John W. Robertson, M.A., B.Sc., Mathematical Master in Robert Gordon's College, Aberdeen, and my colleague, Mr. R. W. Evans, B.A., of Trinity College, Cambridge, for assistance in reading the proof-sheets.

J. M.

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ANALYTICAL GEOMETRY

CHAPTER I

CO-ORDINATES : THE DISTANCE BETWEEN TWO POINTS : GRAPHS

I. *Historical.*—The remotest source from which geometry has reached us is the ancient Egyptians through the work of a priest called Ahmes, entitled *Directions for knowing all Dark Things*. The geometry of the Egyptians sprang from the practical conditions of their life, and made but little progress amongst them. With them it was scarcely more than the art of mensuration, a rough system of which was essential, as the annual overflowing of the Nile obliterated landmarks, and necessitated a survey of the country after the subsidence of the waters. As a branch of science geometry begins among the Greeks with Thales of Miletus, who flourished between 640 and 542 B.C. Its development through many centuries by that brilliant and gifted race we cannot here trace, but one name we must mention in passing, that of Euclid, as his influence is felt by every schoolboy who reaches a stage where geometry is taught. Euclid was attached to the great University of Alexandria, founded by the first of the Ptolemy Pharaohs who placed him in charge of its mathematical school, which remained the most famous in the world till the destruction of the city by the Arabs in 641 A.D. Eminent as a mathematician, Euclid is even more distinguished as an editor and text-book writer. Until lately his *Elements* were widely in use in

secondary schools as a first book on geometry, and more modern books are merely modifications of his work.

The Greek era in mathematics ended with the fall of Alexandria. After that time the study of geometry was almost neglected, and no advance worthy of mention was made till the beginning of the seventeenth century, when Kepler and Desargues laid the foundations of modern pure geometry. With this aspect of the subject we are not here concerned. It differs from analytical geometry, which is to be our study, in the mode by which it investigates the properties of space. Though less elegant, the methods of analytical geometry are far more powerful as a whole, than those of pure geometry, where each problem is solved in a special manner. In analytical geometry, on the other hand, there is one general plan of attack, and it is applicable to many different questions. To a Frenchman named Descartes is due the introduction of analytical geometry, which has given to mathematicians one of their most potent means of investigation. Descartes was the *first to perceive that a point could be fixed on a plane by the* help of two axes, and that the laws of algebra could then be applied to the solution of geometrical problems. Living between 1596 and 1650 he was contemporaneous with many famous men, among them Galileo. The system of co-ordinates familiar to every one who has drawn graphs is that introduced by Descartes. His invention heralds the age of modern mathematics, and the co-ordinates which he used are in honour of him called " Cartesian Co-ordinates."

II. *The Sense of a Line.*—In analytical geometry it is necessary to distinguish between the line LB and the line BL.

L ——————————— B

The distinction is that between a journey from " London to Bristol " and one from " Bristol to London," and is called the sense of the line.

Thus LB denotes not merely the length of the line, but also the mode of traversing it.

The differentiation is effected by agreeing to mark LB with the sign $+$, and BL with the sign $-$ or *vice versâ*.

Thus $LB = -BL$.

III. *Co-ordinates.*—Take a piece of squared paper and draw on it two perpendicular lines, XX' and YY', intersecting at O as shown in the diagram following.

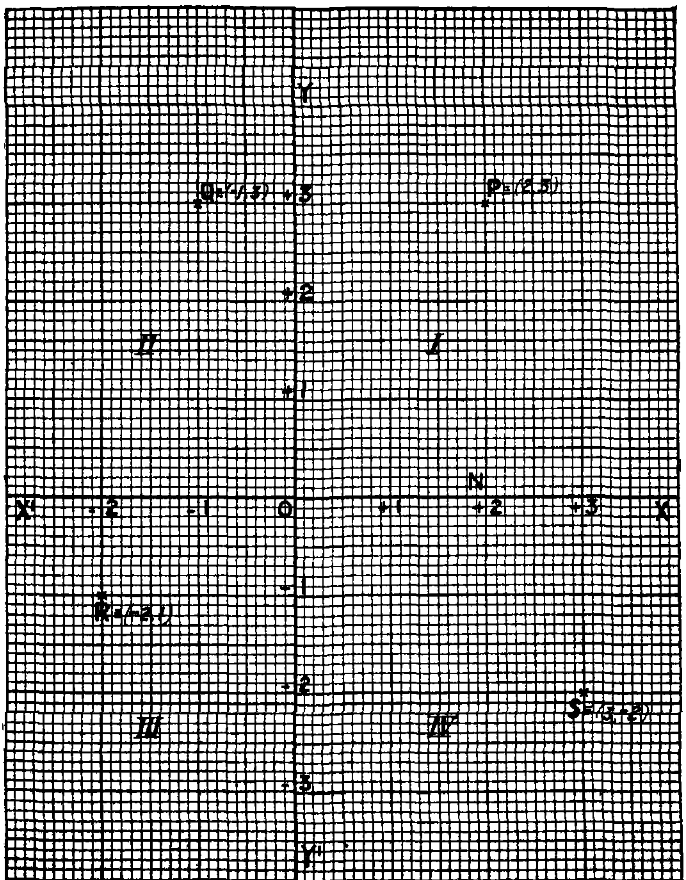


DIAGRAM 1.

O is called the origin, XX' and YY' the axes. The axes divide the plane of the paper into four regions called the four quadrants, which we have numbered I., II., III., and IV.

Let the unit of length contain 10 of the small divisions on the squared paper.

Starting from O count 2 units to the right along OX and then 3 units upwards. Mark the point arrived at P.

Next trace 2 units to the left along OX', and then 1 unit downwards, naming the point now reached R. It is in the third quadrant.

The axes have thus enabled us to give directions for the marking, or as it is called, "plotting," of two points P and R.

It is necessary, however, that we should be able to give these instructions more expeditiously. The following conventions are therefore adopted :—

(i.) Lengths which have the same sense as $X'X$ will be marked $+$, those having the sense of XX' will be marked $-$.

(ii.) Lengths having the same sense as $Y'Y$ will be marked $+$, and these $-$ which have the sense of YY'.

XX' is called the axis of x and YY' the axis of y. The x-axis is divided into two similarly graduated "measuring-rods" OX and OX'. The division numbers on the first will have the sign $+$, as the sense of OX is positive, while the numbers on OX' will have the sign $-$, since the sense of OX' is negative. For the same reasons the scale numbers on OY will be marked plus, and these on OY' will be marked minus.

If from the point P in the diagram, we draw PN perpendicular to XX', then ON is called the "abscissa" or x-co-ordinate of P, and NP its "ordinate" or y-co-ordinate.

Thus the co-ordinates of a point are its distances from the two axes, attention being paid to their signs as we are about to describe.

If we wish to reach the point P on the paper we say "plot the point (2, 3)" meaning, count 2 units along OX, then 3 units upwards, and mark the point so obtained. Within

the brackets the abscissa is always named first and the ordinate second. The point R would therefore be the point $(-2, -1)$, since to arrive at it we must count 2 units along the negative scale OX' and then 1 unit downwards, or in the negative direction for ordinates.

The following points are situated one in each quadrant, as shown in the diagram.

$$P \equiv (2, 3),\ Q \equiv (-1, 3),\ R \equiv (-2, -1) \text{ and } S \equiv (3, -2).$$

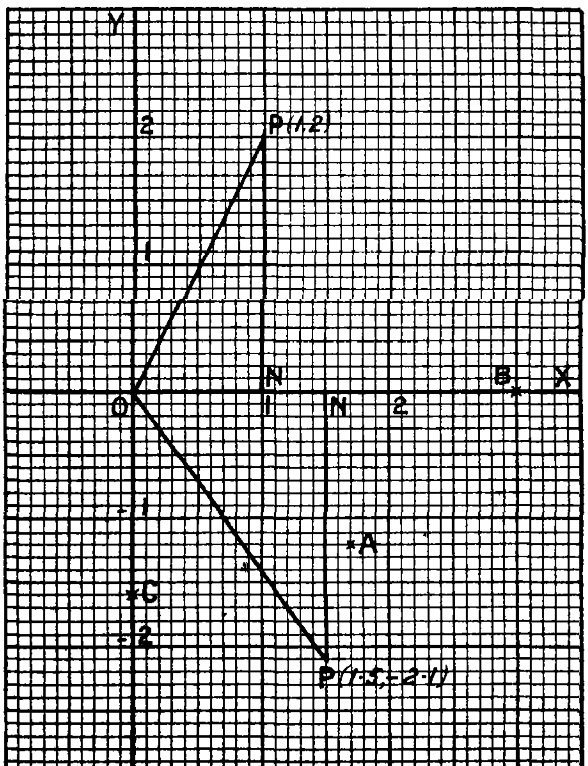


DIAGRAM 2.

In general if x be the abscissa of a point and y its ordinate, we use the notation $P \equiv (x, y)$.

EXAMPLE 1.—*Plot the following points* $A \equiv (1 \cdot 7, -1 \cdot 2)$ $B \equiv (3, 0)$, *and* $C \equiv (0, -1 \cdot 6)$.

A is in the fourth quadrant, B is on the x-axis and C on the y-axis. (See Diagram 2.)

EXAMPLE 2.—*Plot the point* $P \equiv (1, 2)$ *and find its distance from the origin.*

From P draw PN perpendicular to XX', and join OP. (See Diagram 2, page 5.)

Then
$$\begin{aligned} OP^2 &= ON^2 + NP^2 \\ &= 1^2 + 2^2 \\ &= 5. \\ \therefore OP &= \sqrt{5}. \end{aligned}$$

The length of OP is very nearly $2 \cdot 24$ units.

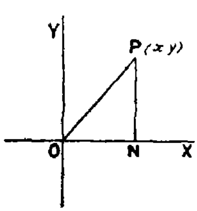
EXAMPLE 3.—*Find the distance of the point* $(1 \cdot 5, -2 \cdot 1)$ *from the origin.*

Make the same construction as before. (See Diagram 2.)
$$\begin{aligned} \therefore OP^2 &= (1 \cdot 5)^2 + (-2 \cdot 1)^2. \\ &= 6 \cdot 66. \\ \therefore OP &= 2 \cdot 58 \text{ nearly.} \end{aligned}$$

EXAMPLE 4.—*In Example 3 what is the size of* $X\hat{O}P$?

We have
$$\begin{aligned} \tan X\hat{O}P &= \frac{NP}{ON} \\ &= \frac{-2 \cdot 1}{1 \cdot 5} \\ &= -1 \cdot 4. \\ \therefore X\hat{O}P &= 305^\circ \; 32'. \end{aligned}$$

IV. *Distance of the point* $P \equiv (x, y)$ *from the origin.*



Draw PN perpendicular to OX.

Join OP.

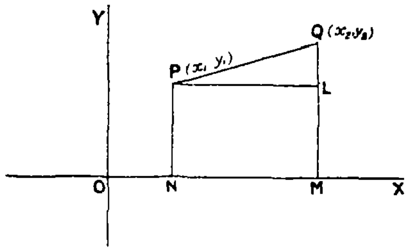
Then
$$\begin{aligned} OP^2 &= ON^2 + NP^2 \\ &= x^2 + y^2. \end{aligned}$$

Let $OP = d_0$
$$\therefore d_0 = \sqrt{x^2 + y^2}.$$

EXAMPLE.—*Find the distance of the point* $(-3, -4)$ *from the origin.*

We have
$$\begin{aligned} d_0^2 &= x^2 + y^2 \\ &= (-3)^2 + (-4)^2 \\ &= 25. \\ \therefore d_0 &= 5. \end{aligned}$$

V. ***Find the length of the line joining the points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$.***



Plot the points P and Q, and draw PN and QM perpendicular to XX'. Draw also PL parallel to it.

Then $$PQ^2 = PL^2 + LQ^2.$$

Now $$PL = NM = OM - ON = x_2 - x_1,$$

and $$LQ = MQ - ML = MQ - NP = y_2 - y_1,$$

$$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Let $PQ = d$

$$\therefore\ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1.—*Find the length of the line joining the points* (1, 1 *and* (3, 2).

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (3 - 1)^2 + (2 - 1)^2 \\ &= 4 + 1 \\ &= 5. \\ \therefore\ d &= \sqrt{5} = 2{\cdot}24 \text{ nearly.} \end{aligned}$$

EXAMPLE 2.—*Find the distance between the points* $(2{\cdot}2, -1{\cdot}4)$ *and* $(-3{\cdot}8, 1{\cdot}6)$.

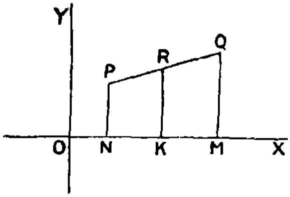
We have
$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-3{\cdot}8 - 2{\cdot}2)^2 + (1{\cdot}6 + 1{\cdot}4)^2 \\ &= (-6)^2 + (3)^2 \\ &= 45. \\ \therefore\ d &= \sqrt{45} = 6{\cdot}7 \text{ nearly.} \end{aligned}$$

VI. ***Find the co-ordinates of the mid-point of the line joining the points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$.***

Let $R \equiv (x, y)$ be the mid-point of PQ.

Draw PN, QM, and RK perpendicular to XX'.

Then since R is the mid-point of PQ, therefore K is the mid-point of NM.



We have $OK = ON + NK$,

and $OK = OM - KM$.

But $NK = KM$

$$\therefore 2OK = ON + OM,$$

$$\therefore OK = \frac{ON + OM}{2}$$

$$\therefore x = \frac{x_1 + x_2}{2}$$

Similarly if perpendiculars be drawn to the y-axis we can show that $y = \frac{y_1 + y_2}{2}$

Hence $$R \equiv (x, y) \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 1.—*Find the co-ordinates of the mid-point of the join of* (1, 3) *and* (5, 2).

$$x = \frac{x_1 + x_2}{2} = \frac{1+5}{2} = 3$$

$$y = \frac{y_1 + y_2}{2} = \frac{3+2}{2} = 2 5.$$

The mid-point is (3, 2·5).

Example 2.—*Find the co-ordinates of the mid-point of the line joining the points* (- 2, 4) *and* (6, - 7).

$$x = \frac{x_1 + x_2}{2} = \frac{-2+6}{2} = 2$$

$$y = \frac{y_1 + y_2}{2} = \frac{4-7}{2} = -1 5.$$

The mid point is (2, - 1·5).

VII. *Loci.*—If a point is constrained to move according to a prescribed law, the path it follows is called its locus. Fix, for example, one end of a piece of string by a tack to a flat piece of cardboard. To the other end fasten a pencil and move the latter so as to keep the string taut. The point of the pencil will leave a circular trace, showing the path along which it has moved. Again fix the two ends of a piece of string to

the cardboard so as to leave the string slack. Now pull it tight by a pencil and move the latter so as always to keep the string in this condition. The trace of the pencil-point is called an ellipse. It is the locus of a point which moves so that the sum of its distances from two fixed points is constant. Other examples could be given, but it is not often easy to trace a locus by mechanical methods.

In analytical geometry points are constrained to move so that their co-ordinates satisfy some assigned equation, as, for example, $y = 3x - 1$ or $x^2 - y^2 = 4$.

The loci can be traced by the help of squared paper, and their nature ascertained. In any equation no more than two unknowns, or variables as they are properly called, must appear, namely the co-ordinates x and y of the moving point. Any other letters in the equation must stand for assigned numbers. The curve over which the point moves is called its locus. It is also called the graph of the equation.

In analytical geometry we name a curve by its equation. Thus we speak of the straight line $2y = x + 3$, or the circle $x^2 + y^2 = 4$ when we mean the graphs of these equations.

EXAMPLE 1.—*A point moves so that its co-ordinates satisfy the equation $y = x$. Find its locus.*

From the equation we have

$$\begin{aligned} y &= -2 \text{ when } x = -2 \\ y &= -1 \text{ when } x = -1 \\ y &= 0 \text{ when } x = 0, \end{aligned}$$

and so on. These results are tabulated in a more compact form, as in the following chart:

x	-2	-1	0	1	2	3
y	-2	-1	0	1	2	3

Any value of y is written below the value of x to which it corresponds.

The scheme will be familiar to any one who has drawn graphs.

The points lie on a straight line through the origin as shown in Diagram 3.

We infer that the locus is a straight line passing through O.

EXAMPLE 2.—*A point moves so that its ordinate is always 4 units long. Find its locus.*

Let $P \equiv (x, y)$ be any position of the variable point. Then $y = 4$

no matter what the value of x may be. We therefore have the following graph chart:

x	-4	-3	-2	-1	0	1	2	3	4
y	4	4	4	4	4	4	4	4	4

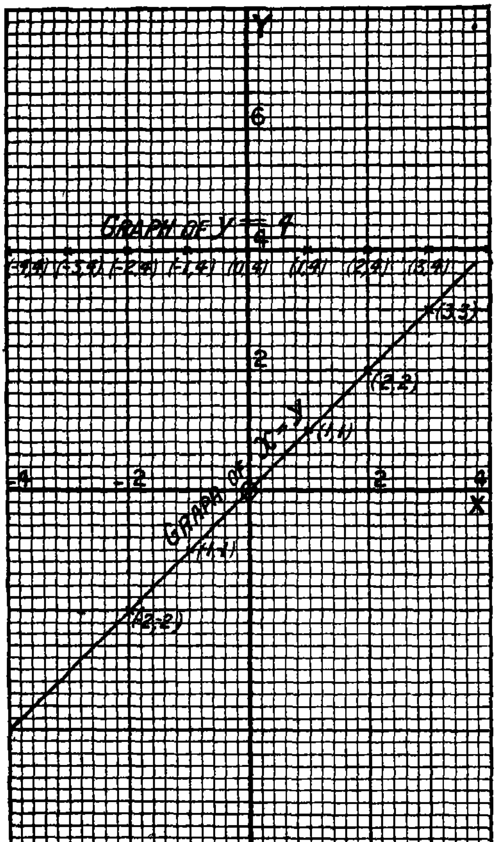


Diagram 3.

These points, as we see from the graph, lie on a straight line parallel to the x-axis. This line is the locus of P. (See Diagram 3.)

EXAMPLE 3.—*A point moves so that its co-ordinates satisfy the equation* $y = \frac{x^2}{2}$. *Draw its locus.*

From the equation $y = \cdot 5$ when $x = 1$
$y = 0$ when $x = 0$,

and so on.

x	-2	$-1{\cdot}5$	-1	$-{\cdot}5$	0	${\cdot}5$	1	$1{\cdot}5$	2
y	2	$1{\cdot}25$	${\cdot}5$	${\cdot}125$	0	${\cdot}125$	${\cdot}5$	$1{\cdot}125$	2

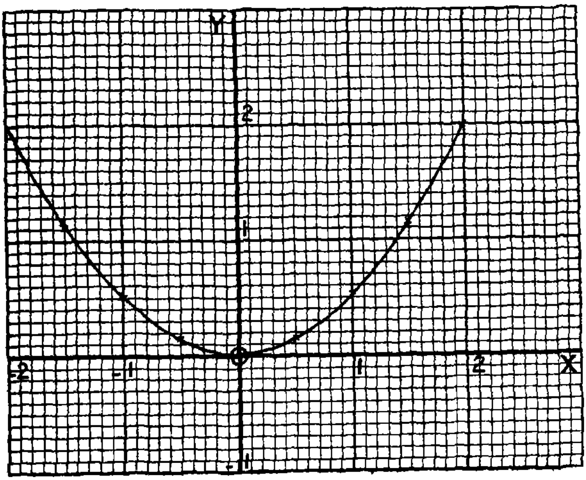


DIAGRAM 4.

The locus is the curve shown in the diagram. It is called a "parabola."

VIII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Plot the points* $A \equiv (2, 4)$, $B \equiv (1, 1)$ *and* $C \equiv (3, 2)$. *Find the area of* $\Delta\ ABC$.

In the following diagram $\Delta\ ABC$ is shown.
Draw the ordinates AL, BM, and CN.

Then $\Delta\ ABC =$ figure $ABMNC -$ trapezium $BMNC$.
$=$ trapez. $ABML +$ trapez. $ALNC -$ trapez. $BMNC$.

Now the area of a trapezium is equal to half the sum of its parallel sides multiplied by the distance between them.

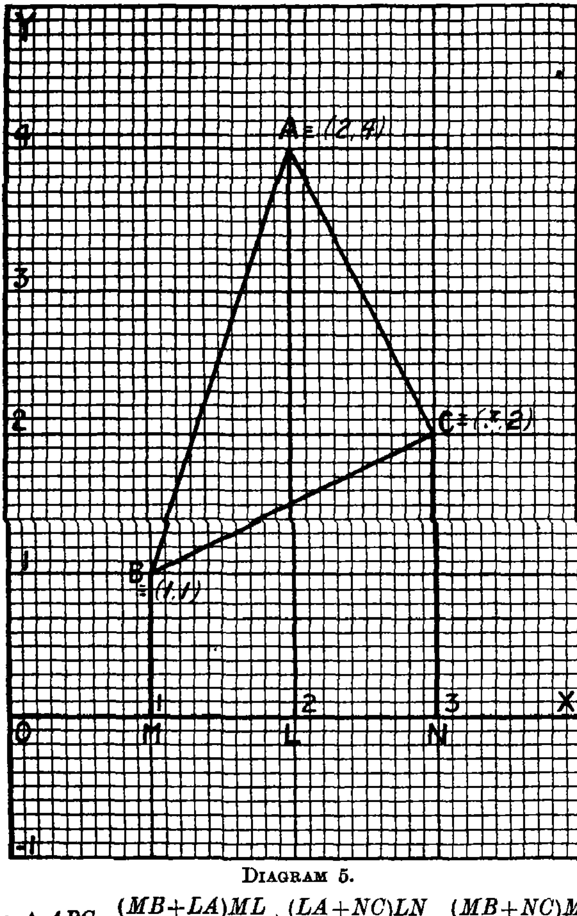


DIAGRAM 5.

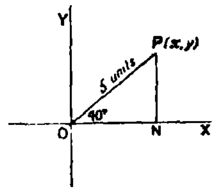
$$\therefore \triangle ABC = \frac{(MB+LA)ML}{2} + \frac{(LA+NC)LN}{2} - \frac{(MB+NC)MN}{2}$$

$$= \frac{1(1+4)+1(4+2)-2(1+2)}{2}$$

$$= \frac{5}{2}.$$

The area of the triangle is 2·5 sq. units.

EXAMPLE 2.—*A point P in the first quadrant is at a distance of* 5 *units from the origin. If the angle XOP is* 40° *find the co-ordinates of P.*



Let $P \equiv (x, y)$. Draw the ordinate NP.

Then $\quad ON = OP \cos 40°$

and $\quad NP = OP \sin 40°$.

$\therefore x = 5 \cos 40° = 3{\cdot}83$ (nearly)

and $\quad y = 5 \sin 40° = 3{\cdot}21$ (nearly).

EXAMPLE 3.—*From a variable point P perpendiculars PN and PM are drawn to the axes. If the perimeter of the rectangle ONPM is always* 6 *units, find the locus of P.*

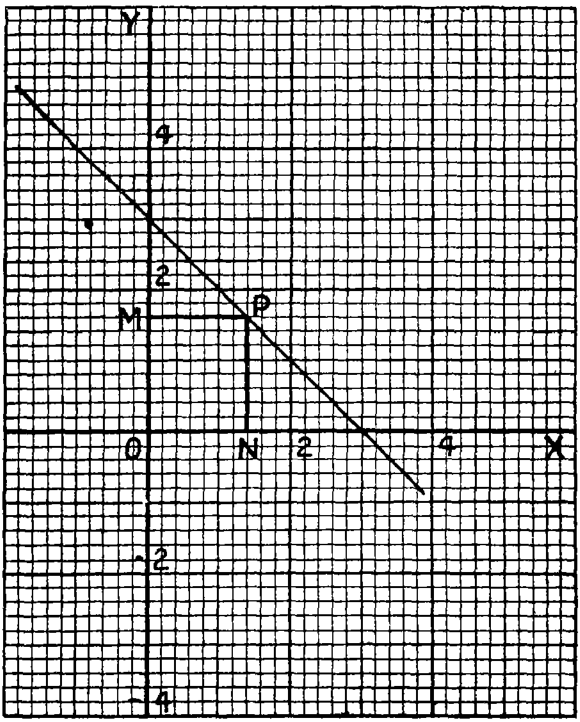


DIAGRAM 6

Let $P \equiv (x, y)$ be any position of the variable point.

Then since $\quad NP + MP = 3$

$\therefore x + y = 3.$

We have to draw the graph of this equation.

x	-2	-1	0	1	2	3	4	5
y	5	4	3	2	1	0	-1	-2

It is a straight line, and is the locus of P. (Diagram 6.)

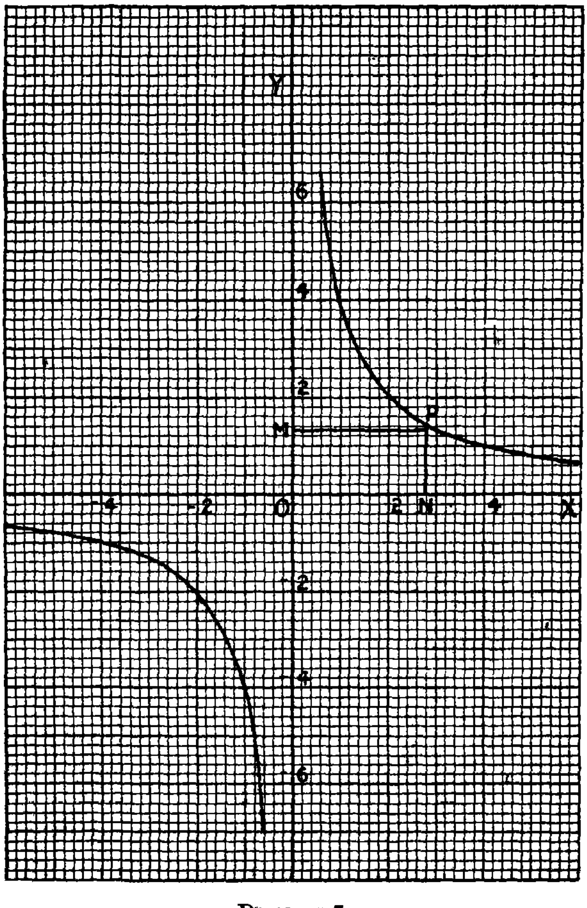


DIAGRAM 7.

Example 4.—*From a variable point P perpendiculars PN and PM are drawn to the axes. If the area of the rectangle $ONPM$ is always 4 sq. units, find the locus of P.*

Let $P \equiv (x, y)$ be any position of the variable point. (Diagram 7.)

Then since $$NP \cdot MP = 4,$$

$$\therefore\ xy = 4.$$

The locus of P is the graph of this equation.

x	-6	-5	-4	-3	-2	-1	$-\cdot 8$	$-\cdot 6$	0	$\cdot 6$	$\cdot 8$	1	2	3	4	5	6
y	$-\cdot 67$	$-\cdot 8$	-1	$-1\cdot 33$	-2	-4	-5	$-6\cdot 67$	∞	$6\cdot 67$	5	4	2	$1\cdot 33$	1	$\cdot 8$	$\cdot 67$

The curve is called a "rectangular hyperbola" and consists of two branches.

RÉSUMÉ

1. The x-co-ordinate of a point is called its "abscissa," the y-co-ordinate is called the "ordinate."

We write $P \equiv (x, y)$.

2. The distance of the point (x, y) from the origin is given by the formula
$$d_0{}^2 = x^2 + y^2.$$

3. The distance between the points (x_1, y_1) and (x_2, y_2) is given by the formula
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

4. The mid-point of the line joining (x_1, y_1) and (x_2, y_2) is
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

5. The graph of an equation is the locus of a point which moves so that its co-ordinates satisfy the equation.

EXAMPLES

1. Plot the following points: $(2, 3)$, $(-4, 1)$, $(-3, -2)$, $(5, -1\cdot 8)$, $(0, -2\cdot 4)$, $(3\cdot 2, 0)$, and $(-2\cdot 9, 0)$.

2. Mark the point $(3, 4)$ and find its distance from the origin.

3. Mark the point $(-1, 3\cdot 3)$ and calculate its distance from the origin.

4. From the point $P \equiv (2, -4\cdot 8)$ perpendiculars PN and PM are drawn to the axes. Calculate the lengths of the diagonals of $ONPM$.

5. A straight line cuts the axes at the points $A \equiv (4, 0)$ and

$B \equiv (0, 1)$. Find the length of AB. Find also the area of $\Delta\, OAB$, and so find the length of the altitude from O correct to two decimal places.

6. Calculate the distances between the following pairs of points $(2, 4)$ and $(5, 6)$, $(-1, 3)$ and $(2, -1)$, $(-3{\cdot}6, 4)$ and $(-5{\cdot}2, -2{\cdot}5)$, $(4{\cdot}5, 0)$ and $(1{\cdot}6, -3{\cdot}8)$ (to the second place of decimals where necessary).

7. Mark the points $P \equiv (1, 2)$ and $Q \equiv (3, 5)$.

Find the length of PQ.

Perpendiculars PN and QM are drawn to XX'.

Find the area of the trapezium $PNMQ$.

8. Draw the triangle whose vertices are $(1, 2)$, $(2, 5)$ and $(4, 3)$.

Obtain its area by the method of Example 1, Article VIII.

Use the same method to find the areas of the triangles whose vertices are—

(i.) $(0, 1{\cdot}6)$, $(2{\cdot}4, 1{\cdot}8)$ and $(3{\cdot}2, 5)$.

(ii.) $(2{\cdot}2, 1{\cdot}4)$, $(5{\cdot}4, 4{\cdot}6)$ and $(3, -2)$.

9. Obtain the area of the first triangle of last example by application of the formula $\sqrt{s(s-a)(s-b)(s-c)}$.

10. Find the co-ordinates of the mid-points of the lines joining the following pairs of points, drawing the figures on squared paper in every case.

(1) $(1, 2)$ and $(3, 4)$.

(2) $(2{\cdot}2, 3{\cdot}4)$ and $(5, 4)$.

(3) $(0, 0)$ and $(-2, 3)$.

(4) $(2{\cdot}6, 0)$ and $(1, 3{\cdot}8)$.

(5) $(-3, 2)$ and $(-1, 4)$.

(6) $(-2, -3{\cdot}8)$ and $(-4{\cdot}6, {\cdot}8)$.

(7) $(0, -2{\cdot}7)$ and $(1{\cdot}7, 4{\cdot}9)$.

(8) $(-3{\cdot}6, -2{\cdot}4)$ and $(3{\cdot}6, 2{\cdot}4)$.

11. Mark the point $P \equiv (4, 2)$. Draw PM and PN perpendicular to the axes. What is the mid-point of MN? Calculate its distance from the origin.

12. In the equation $2x+5y=10$, what is the value of x when $y=0$, and the value of y when $x=0$?

At what points P and Q does the graph of the equation therefore cross the axes?

What are the co-ordinates of the mid-point of PQ?

13. In the equation $x^2+y^2=4$, what are the values of x when $y=0$, and the values of y when $x=0$?

Hence name the points P Q R and S where the graph crosses the axes.

Draw the square $PQRS$ and find the area of the square obtained by joining its mid-points.

14. At what points does the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$ cross the axes?

If P Q R and S are the points, find L and M, the mid-points of PS and RQ, and show that the origin is the mid-point of LM. Find also the area of the rhombus $PQRS$.

15. The graph of $y=x^2-4x+3$ crosses the x-axis at two points A and B and the y-axis at a point C. Find these points and the area of $\Delta\ ABC$.

16. Find the area and the perimeter of the triangle whose vertices are the three points at which the graph of $y=x^2-x-6$ crosses the axes.

17. Find the points A and B at which the graph of $\frac{x}{a}+\frac{y}{b}=1$ crosses the axes.

Find the distance of C, the mid-point of AB from O.
Show that $AB=2OC$.

18. A point $P\equiv(h, k)$ is taken.
It is joined to a point Q on XX' which is such that PQ is bisected by the y-axis.
Find the co-ordinates of Q.

19. P and Q are the points where the graph of $y=x^2+4x-5$ crosses the x-axis, and R is the point where it crosses the y-axis. Calculate the distance between R and the mid-point of PQ.

20. Draw the quadrilateral whose vertices are (0, 0), (5, 0), (2, 4), and (0, 6). Calculate its area.
(Hint: Divide the quadrilateral into two triangles by the diagonal through O.)

21. A and B are the points where the graph of $2x+y-4=0$ crosses the axes. C is the point (6, 9). Obtain the area of the quadrilateral $OACB$.

22. If C is the circumference of a circle, A its area, and r its radius, then we know that

$$C=2\pi r$$
$$\text{and } A=\pi r^2.$$

Draw the graphs of these equations on the same sheet of squared paper, plotting values of r along the horizontal axis.
What is the radius of the circle whose area is numerically equal to its circumference?

23. A point lies in the third quadrant at a distance of 2 units from the origin. If the length of its ordinate is 1·6 units, what is the length of its abscissa?

24. Plot a point P in the first quadrant at a distance of 3·4 units from the origin and such that $X\hat{O}P=35°$. Calculate the lengths of the co-ordinates correct to the second place of decimals.

25. A man walks from a place in a direction 28° N. of E. at the rate of $3\frac{1}{2}$ m. p.h. After 2 hr. 20 min. tell (i.) how far north and (ii.) how far east he is of the place from which he started.

26. The point $(-1, 4)$ is the mid-point of a line, one of whose extremities is at the point $(-5, 6)$. Where is the other extremity?

27. The support of a see-saw has a height of 3 ft. Show that if one end of the swing-plank is 1 ft. above the ground, the height of the other end in no way depends on the length of the plank.

28. A point moves so that its co-ordinates satisfy the equation $x=y$. Draw its locus.

29. Draw the locus of a point which moves so that its co-ordinates satisfy the equation $y=x-3$.

30. Draw the loci of the points which move so that their co-ordinates satisfy the following equations:—

(1) $2x+3y=6$ (3) $x^2+y^2=8$

(2) $y=\dfrac{x^2}{10}$ (4) $\dfrac{x^2}{9}-\dfrac{y^2}{3}=1.$

31. From a variable point P perpendiculars PM and PN are drawn to the axes. If $P\equiv(x, y)$, and if the area of the rectangle $ONPM$ is always 5 square units, express the equation connecting the co-ordinates. Use it to draw the locus of P.

32. From a variable point P perpendiculars PM and PN are drawn to the axes. If the diagonal of the rectangle $ONPM$ is always 3 units long, express this in an equation and use it to draw the locus of P.

33. B is a fixed point on the y-axis such that OB is 10 units long. P is a variable point, the square on whose distance from XX' is equal in area to the triangle OPB. Draw the graph of the equation which gives the locus of P.

34. A circle is described with centre O and radius 2 units. P is a variable point and PN is perpendicular to XX'. If the tangent from N to the circle is always equal to NP, obtain the equation giving the locus of P, and draw its graph.

35. A square $ABCD$ is described on the line joining the points $A\equiv(6, 0)$ and $B\equiv(0, 4)$ so as to be turned away from O. Find the co-ordinates of C and D and of the centre of the square.

Deduce from the results that the centre of the square lies on the bisector of $X\hat{O}Y$.

CHAPTER II

THE GENERAL EQUATION OF THE FIRST DEGREE IN x AND y: EQUATION TO A STRAIGHT LINE IN "INTERCEPT FORM": THE INTERSECTION OF TWO STRAIGHT LINES

I. *The equation to a locus.*

We shall indicate very particularly the procedure followed in analytical geometry in dealing with a locus problem.

(i.) The law of constraint under which the variable point moves is carefully stated.

(ii.) Suitable axes are chosen and the equation connecting the co-ordinates is worked out.

(iii.) The graph of the equation is drawn.

This graph is the required locus, for the co-ordinates of every point on it satisfy the equation which algebraically expresses the law of constraint under which the variable point moves. The equation itself must contain no variables other than the co-ordinates. The rest of it consists of constants or numbers which when assigned or determined are fixed once and for all, otherwise a graph could not be drawn. Take, for example, the equation $2x^2-5xy+3=0$. The constants appearing in it are $2, -5$, and 3, and these cannot be changed, for if we did so we would proceed to draw a portion of another graph. It is only the values of x and y which vary, and of those only one at a time is at the disposal of the worker, for when a value is assigned to x say, then the value of y is given by the equation. The variable or co-ordinate to which we are assigning values

is called the independent variable, and the other is called the dependent variable. In our equations we shall often meet other letters, however, besides x and y, but these letters stand for given numbers, and, as we have said, are called constants. For example, the equation we have just been using might be looked on as a particular member of a general type $lx^2+mxy+n=0$, the various members being obtained by stating values for l, m, and n, as 2, -5, and 3 in the example chosen, and we must know them before we start to draw the graph.

Certain graphs are so important that we learn to recognise them from their mere equations. One of these is the straight line. It will form the subject of discussion in the present chapter.

Such an equation as $3x-5y+2=0$ is called an equation of the first degree in x and y. No term contains x or y to a higher power than the first, nor any product of these variables. For example, the equations $3x^2-5x+4=0$ and $6xy+3x-5y=0$ are not of the first degree in x and y, as the former contains a term in x^2 and the latter one in xy. The equation $3x-2=0$ may, however, be regarded as an equation of the first degree in x and y, where the term in y is absent owing to its coefficient being zero. In full the equation would be $3x+0y-2=0$. We shall prove that all equations of the first degree in x and y have straight lines for their graphs. Hence they are sometimes called linear equations. Their general type is $lx+my+n=0$. The term n is called the absolute term.

II. *Introductory example: Points M and L are taken on the x and y axes respectively, so that $OM=4$ and $OL=3$ units. P is a variable point, such that the quadrilateral OMPL is always 9 sq. units. Find the locus of P.*

Since $\qquad OM=4$ and $OL=3$,

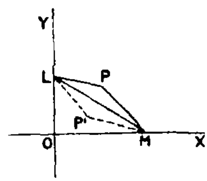
$$\therefore LM=5 \text{ units},$$

$$\text{and } \Delta\, OML=6 \text{ sq. units.}$$

Subtracting its area from that of the whole quadrilateral $OMPL$, we have

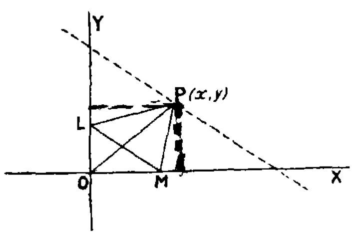
$$\Delta PML = 3 \text{ sq. units.}$$

Now this triangle stands on a fixed base LM of length 5 units. Hence P moves along a line parallel to LM and at a distance of 1·2 units from it.



Without reflection we are apt to say that the locus of P is two lines parallel to LM lying on either side of it and at a distance of 1·2 units from it. But in this example P cannot lie on the origin side of LM as at P', for in that case the quadrilateral $OMP'L = \Delta OML - \Delta P'ML = 6 - 3 = 3$, whereas its area was given to be 9 sq. units. The quadrilateral therefore prevents the locus from being an ambiguous one.

III. *The graph of any equation $lx + my + n = 0$, of the first degree in x and y is a straight line.*



Let $P \equiv (x, y)$ be any point in the plane of the axes.

Take M on OX and L on OY so that $OM = m$ and $OL = l$.

Then $$lx = OL \times x = 2\Delta\, OPL,$$

$$\text{and } my = OM \times y = 2\Delta\, OPM.$$

$$\therefore lx + my = 2 \text{ quad. } OMPL,$$

a geometrical interpretation of $lx + my$ which should be noted as it will be used hereafter.

It therefore follows that if P is on the graph of

$$lx + my + n = 0,$$

then $$2 \text{ quad. } OMLP + n = 0,$$

$$\therefore \text{quad. } OMLP = -\frac{n}{2}$$

$$= \text{constant.}$$

Now part of this quadrilateral, namely $\Delta\, OML$, is fixed.

Hence the remainder $\Delta\, PLM$ is of constant area.

Its base LM is fixed, so that the locus of its vertex P is a straight line parallel to LM. But the locus of P is the graph of the equation $lx+my+n=0$. The latter is therefore a straight line.

In passing we shall point out an immediate corollary which,

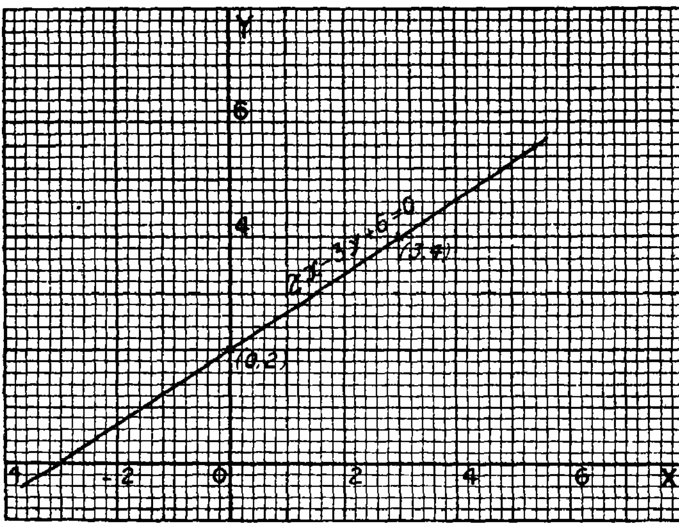


DIAGRAM 1

however, will be fully dealt with in a subsequent chapter. Suppose that l and m retain the same values throughout, but let n range through a series of values, as for example $lx+my-1=0$, $lx+my-3=0$, $lx+my+9=0$, and so on. Then we shall evidently obtain as the graphs of these equations a series of straight lines parallel to LM. Hence the graphs of linear equations which differ only in the absolute term, are parallel straight lines.

EXAMPLE 1.—*Draw the graph of the equation* $2x-3y+6=0$.

We know that the graph is a straight line since the equation is of the first degree in x and y. We need therefore plot only two points on it.

Now $$y=4 \text{ when } x=3,$$ $$\text{and } y=2 \text{ when } x=0.$$

The line is that joining the points (3, 4) and (0, 2). (Diagram 1.)

EXAMPLE 2.—*From a variable point P, perpendiculars PN and PM are drawn to the axes. If the length of the rectangle $ONPM$ always exceeds the breadth by 2 units, find the locus of P.*

Let $P\equiv(x, y)$.

Then since $ON-NP=2$, $\therefore x-y=2$.

The locus of P is the graph of this equation, and is a straight line. By plotting the points (3, 1) and (1, −1) which lie on the graph, we obtain the straight line. (Diagram 2.)

IV. *Particular cases of the equation* $lx+my+n=0$.

(i.) First let us suppose that $n=0$ but that l and m are not zero.

The equation then reduces to $lx+my=0$.

It is evidently satisfied by the values $x=0$ and $y=0$.

Hence the graph is a straight line passing through the origin.

EXAMPLE.—*Draw the graph of* $y=\frac{1}{2}x$. (See Diagram 3.)

The graph is a straight line, so we shall find two points on it.

Now $$y=0 \text{ when } x=0,$$ $$\text{and } y=1 \text{ when } x=2.$$

The line therefore passes through the origin and the point (2, 1).

(ii.) Let $l=0$, but let m and n not be zero.

The equation then becomes

$$my+n=0,$$
$$\text{or } y=-\frac{n}{m}.$$

The ordinates of all points on this graph are therefore equal no matter what values the abscissae may have.

x	-3	-2	-1	0	1	2	3	4
y	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$	$-\frac{n}{m}$

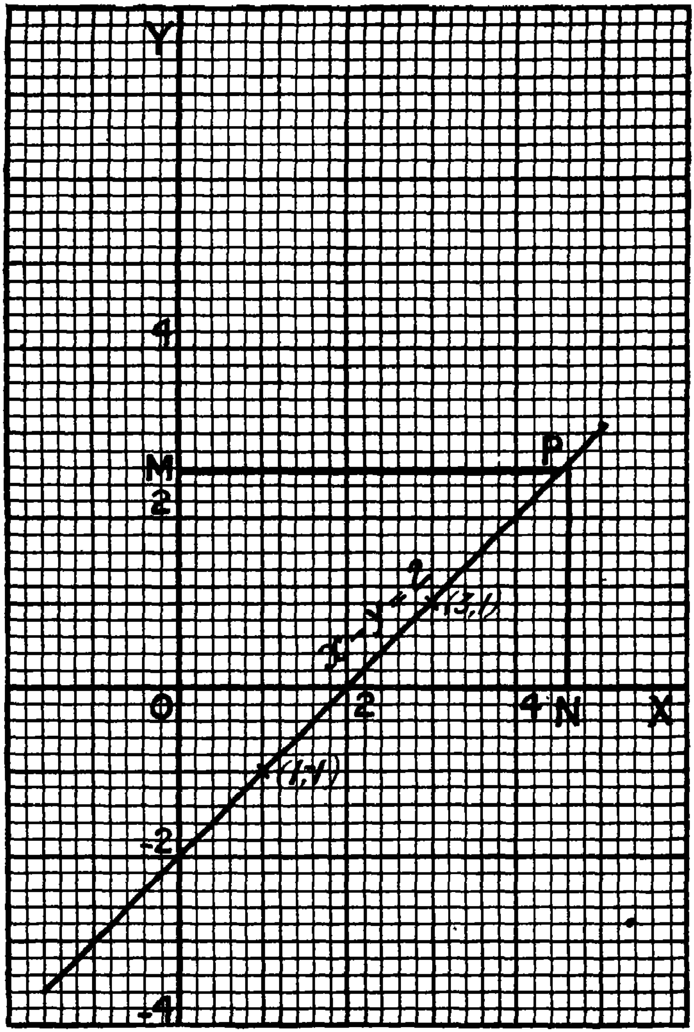


DIAGRAM 2.

The graph is a straight line parallel to the x-axis, and at a distance of $\frac{n}{m}$ units from it.

The side of XX' on which the line lies will depend on the sign of $\frac{n}{m}$.

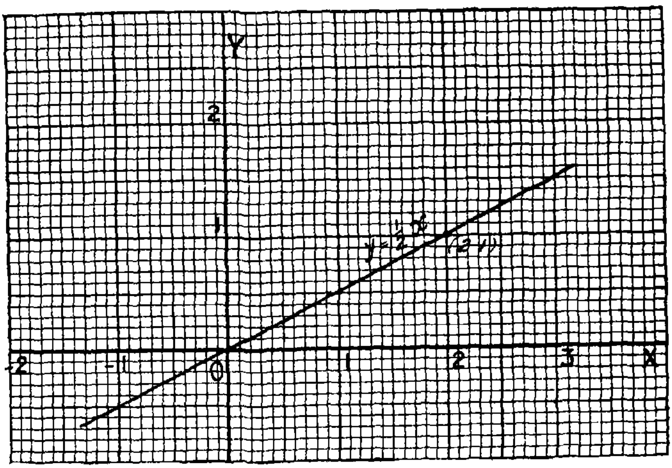


DIAGRAM 3.

EXAMPLE.—*Draw the graph of the equation* $y = 3$.

x	-2	-1	0	1	2	3
y	3	3	3	3	3	3

It is a straight line parallel to XX' and at a distance of 3 units from it. The line $y = -3$ would lie on the under side of XX', and at the same distance from it as $y = 3$. (Diagram 4.)

(iii.) In a similar manner it can be shown that $lx + n = 0$ is the equation to a straight line parallel to the y-axis.

The equations $y = 0$ and $x = 0$ are evidently those of the x and y axes respectively.

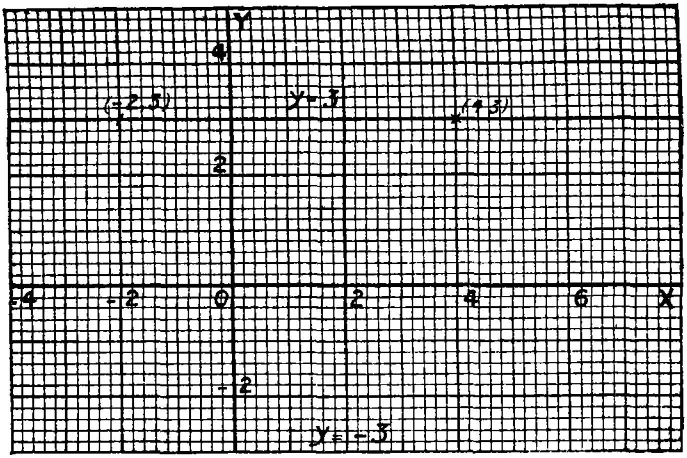


DIAGRAM 4.

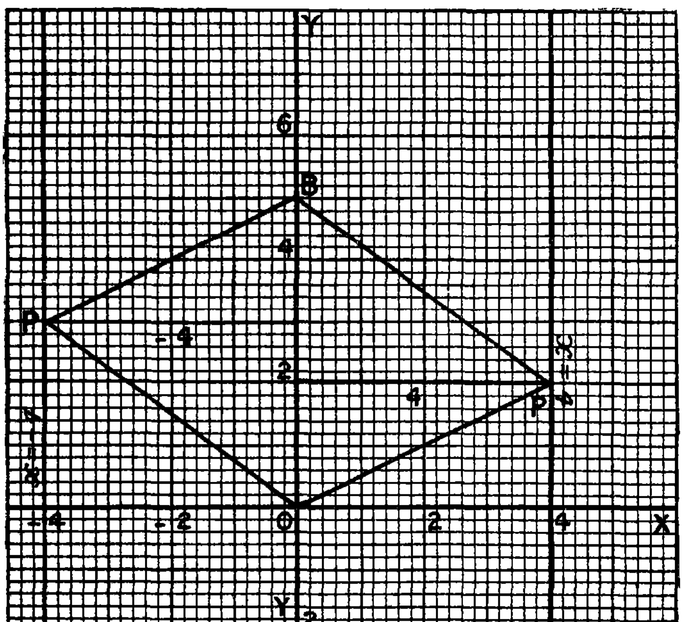


DIAGRAM 5.

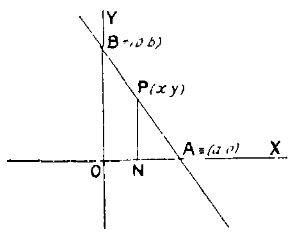
EXAMPLE.—*A length OB of 5 units is taken on the y-axis. A variable point P moves so that the area of ΔOPB is always 10 sq. units. Find the locus of P.*

Since the base of the triangle OPB is 5 units and the area 10 sq. units, the height is therefore 4 units. Hence P moves at a distance of 4 units from the y-axis. It can lie on either side of that line. Its locus is one or other of the straight lines parallel to YY' whose equations are

$$x=4 \text{ and } x=-4. \quad \text{(Diagram 5.)}$$

V. *Equation to a straight line in "Intercept Form."*

We are going to establish the equation to a straight line in a form which will show us immediately where the line crosses the axes. In this shape it is often of great convenience.



Let the straight line cut XX' at A and YY' at B, and let $OA=a$ and $OB=b$.

OA and OB are called the intercepts of the line on the axes.

The line crosses the axes at the points (a, o) and (o, b).

Take $P\equiv(x, y)$ any point on the line, and draw PN perpendicular to XX'.

Then by similar triangles OAB and NAP we have

$$\frac{NP}{OB}=\frac{NA}{OA}$$

$$=\frac{OA-ON}{OA},$$

$$\therefore \frac{y}{b}=\frac{a-x}{a}$$

$$=1-\frac{x}{a},$$

$$\therefore \frac{x}{a}+\frac{y}{b}=1.$$

Note that the right-hand member of this equation is $+1$, and that the reciprocals of the coefficients of x and y are the intercepts made on the respective axes.

EXAMPLE 1.—*What are the intercepts made on the axes by the graph of* $\frac{x}{4}+\frac{y}{5}=1$?

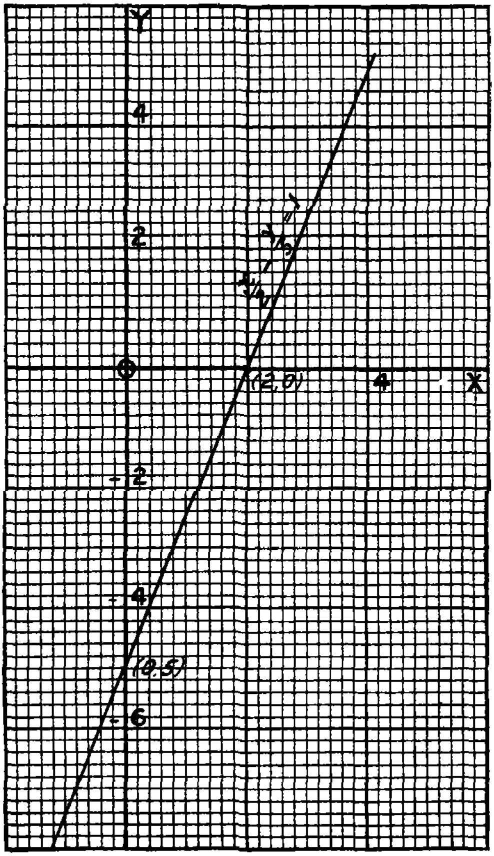


DIAGRAM 6.

On comparing this equation with the standard one we see that $a=4$ and $b=5$.

The line makes positive intercepts of 4 units on the x-axis and 5 units on the y-axis.

EXAMPLE 2.—*What intercepts does the graph of* $\frac{x}{2}-\frac{y}{5}=1$ *make on the axes?*

In this case $a=2$ and $b=-5$.

Hence the line makes a positive intercept of 2 units on the x-axis, and a negative one of 5 units on the y-axis. In other words it crosses the axes at the points (2, 0) and (0, −5). (See Diagram 6.)

This can easily be verified from first principles. For at the point where the graph crosses XX' we have $y=0$. The equation gives the corresponding value of x as 2. Again at the point of intersection with the y-axis we have $x=0$. The value of y corresponding is −5.

EXAMPLE 3.—*Throw the equation* $2x-4y=3$ *into intercept form and so find where its graph crosses the axes.*

Divide by 3 in order that the right-hand member of the equation may be +1,

$$\therefore \tfrac{2}{3}x-\tfrac{4}{3}y=1,$$

$$\text{or } \frac{x}{3/2}-\frac{y}{3/4}=1.$$

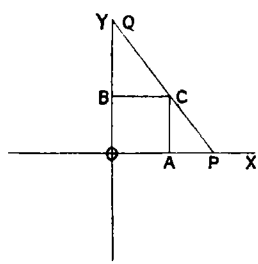
Thus we see that $a=\frac{3}{2}$ and $b=-\frac{3}{4}$.

The graph crosses the axes at the points $(\frac{3}{2}, 0)$ and $(0, -\frac{3}{4})$.

As before we could obtain these results from first principles, since the original equation shows us that $y=0$ when $x=\frac{3}{2}$ and $x=0$ when $y=-\frac{3}{4}$.

x	1·5	0
y	0	−·75

EXAMPLE 4.—*Lengths OA and OB are marked off along OX and OY such that* $OA=OB=1$. *The square OACB is completed. Through C any line is drawn cutting X′X at P and Y′Y at Q. Prove that* $\frac{1}{OP}+\frac{1}{OQ}=1$.



Let the straight line PQ make intercepts of p and q units on the x and y axes respectively. Then its equation will be

$$\frac{x}{p}+\frac{y}{q}=1.$$

But $C\equiv(1, 1)$ is a point on the line.

$$\therefore \frac{1}{p}+\frac{1}{q}=1,$$

$$\therefore \frac{1}{OP}+\frac{1}{OQ}=1.$$

VI. *Determination of equations to straight lines from given data.*

In Article II. of this chapter we stated that the general equation of the first degree in x and y was

$$lx + my + n = 0,$$

where l, m, and n were assigned numbers such as $3, -2, -5$.

At first sight it would seem that there were three constants, l, m, and n, to be determined before we could have the equation to a particular line. This is not so, however, as we can divide both sides by any one of them, say n, obtaining the form

$$\frac{l}{n}x + \frac{m}{n}y + 1 = 0.$$

We now see that there are only two constants to be determined, namely, the ratios $\frac{l}{n}$ and $\frac{m}{n}$.

Let $\frac{l}{n} = \alpha$ and $\frac{m}{n} = \beta$.

The equation then is

$$\alpha x + \beta y + 1 = 0,$$

which clearly shows only two constants.

Compare this with the number of constants in the intercept form $\frac{x}{a} + \frac{y}{b} = 1$.

Corollary.—A straight line can be made to satisfy two independent conditions, since its equation contains two independent constants.

The following examples illustrate this.

Example 1.—*Find the equation to the straight line which passes through the points* (1, 2) *and* (3, −2).

Suppose its equation is

$$\alpha x + \beta y + 1 = 0.$$

Since (1, 2) is a point on the line

$$\therefore\ \alpha + 2\beta + 1 = 0.$$

Since (3, −2) is a point on the line

$$\therefore\ 3\alpha - 2\beta + 1 = 0.$$

On solving these equations we have $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{4}$.

The required equation is

$$-\tfrac{1}{2}x - \tfrac{1}{4}y + 1 = 0 \quad . \quad . \quad . \quad . \quad (1),$$

$$\text{or } 2x + y - 4 = 0 \quad . \quad . \quad . \quad . \quad (2).$$

In number (1) of these two forms we see the equation appearing with fractional coefficients as in

$$\frac{l}{n}x + \frac{m}{n}y + 1 = 0,$$

while number (2) shows us the equation cleared of fractions, as in

$$lx + my + n = 0.$$

EXAMPLE 2.—*Find the equation to a straight line which makes a negative intercept of 4 units on the x-axis and passes through the point* (2, 4·5).

Let the required equation be

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Then $a = -4$, making the equation

$$-\frac{x}{4} + \frac{y}{b} = 1.$$

But the point (2, 4·5) lies on the line,

$$\therefore \; -\frac{2}{4} + \frac{4{\cdot}5}{b} = 1,$$

$$\text{whence } b = 3.$$

The equation is therefore

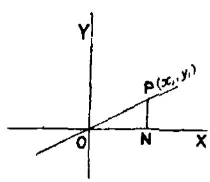
$$-\frac{x}{4} + \frac{y}{3} = 1$$

$$\text{or } 3x - 4y + 12 = 0.$$

EXAMPLE 3.—*Find the equation of the straight line joining the origin to the point* (x_1, y_1).

Let the required equation be

$$lx + my + n = 0.$$



Since the line passes through the origin,

$$\therefore n = 0,$$

$$\therefore lx + my = 0 \text{ or } lx = -my \quad . \quad (1).$$

But (x_1, y_1) is a point on this line,

$$\therefore lx_1 = -my_1 \quad . \quad . \quad (2).$$

Hence by (1) and (2)

$$\frac{lx}{lx_1} = \frac{-my}{-my_1},$$

$$\therefore \frac{x}{x_1} = \frac{y}{y_1}.$$

This is the required equation and it should be noted carefully and remembered.

By putting $x=x_1$ and $y=y_1$ it is easy to verify mentally that it is correct.

VII. *The point of intersection of two straight lines.*

Let the equations to the straight lines be

$$x-2y=-1,$$
$$\text{and } x-3y=-3.$$

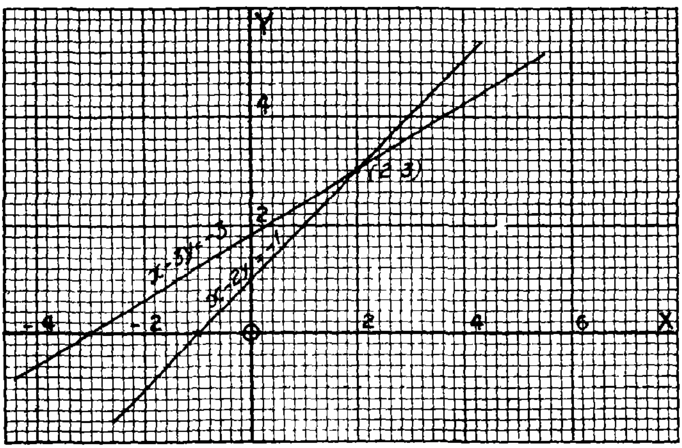


DIAGRAM 7.

Draw their graphs. We find that they cross at the point (2, 3).

We see at once that the values $x=2$ and $y=3$ must appear in the graph chart belonging to each equation. In other words, the values $x=2$ and $y=3$ must satisfy both equations. They are the solutions of the two equations.

The algebraic process of solving these two simple simultaneous equations in x and y is assumed to be familiar to every reader.

Generalising, we state that the co-ordinates of the point of intersection of two straight lines are the values of x and y which satisfy both their equations.

EXAMPLE.—*Where do the straight lines whose equations are* $13x+15y=26$ *and* $13x+25y=52$ *intersect?*

Solution of their equations shows that the straight lines intersect at the point $(-1, 2\cdot6)$.

This result is verified graphically in Diagram 8, where the graphs are shown.

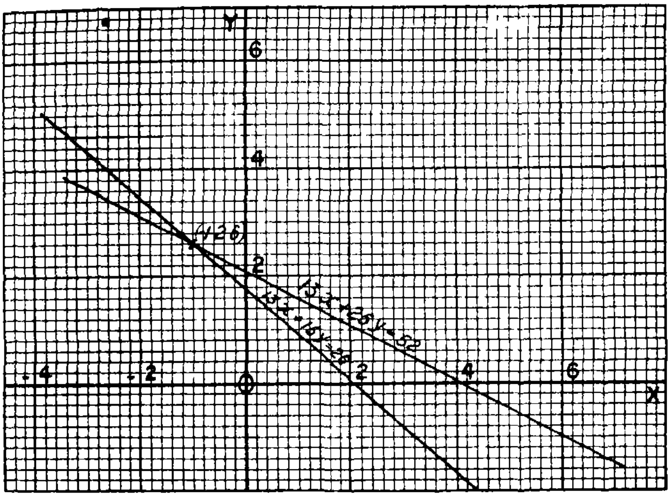


DIAGRAM 8

VIII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Find from first principles where the straight line* $lx+my+n=0$ *crosses the axes and verify the results by throwing the equation into intercept form.*

(i.) Where the graph crosses $X'X$ we have $y=0$.

The value of x corresponding is obtained from the equation.

$$\therefore lx+n=0,$$

$$\therefore x=-\frac{n}{l}.$$

The graph crosses the x-axis at the point $\left(-\frac{n}{l}, 0\right)$

Similarly it crosses $Y'Y$ at a point where $x=0$.

Then from the equation we have

$$my+n=0,$$

$$\therefore y=-\frac{n}{m}$$

The point of intersection with the y-axis is therefore $\left(0, -\frac{n}{m}\right)$.

(ii.) To obtain these results by throwing the equation into intercept form we first write the equation thus—

$$lx+my=-n.$$

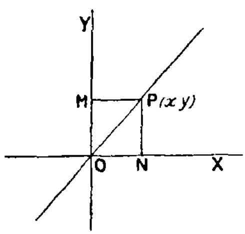
We next divide both sides by $-n$ so that the right-hand member may be $+1$.

$$-\frac{l}{n}x-\frac{m}{n}y=1$$

$$\text{or}\quad \frac{x}{-n/l}+\frac{y}{-n/m}=1.$$

Thus we see that the intercepts on the x and y axes are respectively $-\frac{n}{l}$ and $-\frac{n}{m}$, or, in other words, that the straight line crosses the axes at the points $\left(-\frac{n}{l}, 0\right)$ and $\left(0, -\frac{n}{m}\right)$ as found by method (i.).

Example 2.—*Two lines XX' and YY' are at right angles to one another. A point P moves so that its distance from the former is three times its distance from the latter. Find its locus.*



Take XX' as axis of x and YY' as axis of y.

Let $P\equiv(x, y)$ be any position of the variable point.

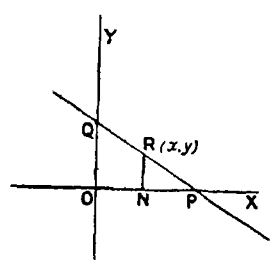
Draw PN perpendicular to XX' and PM to YY'

By hypothesis $NP=3MP$,

$$\therefore\ y=3x.$$

The locus of P is the graph of this equation, and is therefore a straight line passing through O.

Example 3.—*A variable straight line cuts the axes at P and Q so that $OP+OQ$ is constant. Find the locus of its mid-point.*



Let $R\equiv(x, y)$ be the mid-point of any line PQ of the system.

Then $x=\frac{1}{2}OP$, and $y=\frac{1}{2}OQ$,

$$\therefore\ OP=2x \text{ and } OQ=2y.$$

But $OP+OQ=c$ (a constant),

$$\therefore\ 2x+2y=c.$$

The locus of R is therefore the straight line which is the graph of this equation.

EXAMPLE 4.—*A straight line cuts the axes at A and B so that $OA=2$ units and $OB=5$ units. If P is any point on AB find the locus of Q the mid-point of OP.*

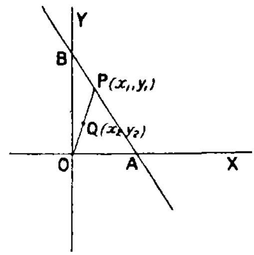
Let $P\equiv(x_1, y_1)$, and $Q\equiv(x_2, y_2)$.

Then since Q is the mid-point of OP,

$$\therefore x_2=\frac{x_1}{2} \text{ and } y_2=\frac{y_1}{2},$$

$$\therefore x_1=2x_2 \text{ and } y_1=2y_2.$$

By Article V. the equation to AB is



$$\frac{x}{2}+\frac{y}{5}=1.$$

Since $P\equiv(x_1, y_1)$ is a point on the line,

$$\therefore \frac{x_1}{2}+\frac{y_1}{5}=1,$$

$$\therefore \frac{2x_2}{2}+\frac{2y_2}{5}=1,$$

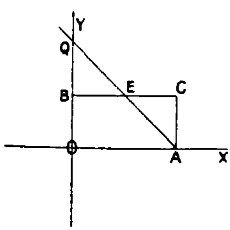
$$\therefore 5x_2+2y_2=5.$$

Hence the locus of $Q\equiv(x_2, y_2)$ is the graph of the equation

$$5x+2y=5.$$

It is a straight line cutting the axes at the points $(1, 0)$ and $(0, \frac{5}{2})$. Its intercepts are half those of AB.

EXAMPLE 5.—*OACB is a rectangle. If E is the mid-point of BC find the equation to AE, taking OA and OB as axes of x and y respectively. Thence find where AE crosses the y-axis.*



Let $OA=a$ and $OB=b$.

Then $C\equiv(a, b)$, so that $E\equiv\left(\frac{a}{2}, b\right)$.

If AE cuts OY at Q, then the equation to AE is

$$\frac{x}{a}+\frac{y}{q}=1,$$

where $q=OQ$.

But $E\equiv\left(\frac{a}{2}, b\right)$ is a point on the line,

$$\therefore \frac{a}{2a}+\frac{b}{q}=1,$$

$$\therefore \frac{b}{q}=\frac{1}{2},$$

$$\therefore q=2b,$$

a result, of course, easily obtained by ordinary geometry.

Hence the equation to AE is

$$\frac{x}{a}+\frac{y}{2b}=1.$$

The line crosses the y-axis at the point $(0, 2b)$.

EXAMPLE 6.—*In last example find where OC and AE intersect.*
Since $C\equiv(a, b)$ the equation to OC is

$$\frac{x}{a}=\frac{y}{b} \text{ (Art. VI. Ex. 3).}$$

We have now to solve the equations

$$\frac{x}{a}+\frac{y}{2b}=1 \quad . \quad . \quad . \quad . \quad (1),$$

$$\text{and } \frac{x}{a}=\frac{y}{b} \quad . \quad . \quad . \quad . \quad (2).$$

Substituting from (2) in (1) for y, we have

$$\frac{x}{a}+\frac{x}{2a}=1,$$

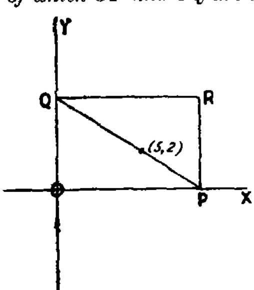
$$\therefore \frac{3x}{2a}=1,$$

$$\therefore x=\frac{2a}{3}.$$

Hence from (2) $\quad y=\frac{2b}{3}.$

Thus AE cuts OC at a point of trisection.

EXAMPLE 7.—*A straight line is drawn through the point (5, 2) and cuts the axes at P and Q. If R is the remaining vertex of the rectangle of which OP and OQ are sides, find the equation to its locus.*



Let $R\equiv(x', y')$.
Then $OP=x'$ and $OQ=y'$.
The equation to PQ is therefore

$$\frac{x}{x'}+\frac{y}{y'}=1.$$

Now (5, 2) is a point on this line,

$$\therefore \frac{5}{x'}+\frac{2}{y'}=1,$$

$$\therefore 2x'+5y'=x'y'.$$

The locus of $R\equiv(x', y')$ is the graph of the equation $xy=2x+5y$.

The graph of the equation can be drawn by plotting a number of points as in the following diagram.

Notice that the curve draws nearer and nearer to the lines $y=2$ and $x=5$, though it never actually reaches them.

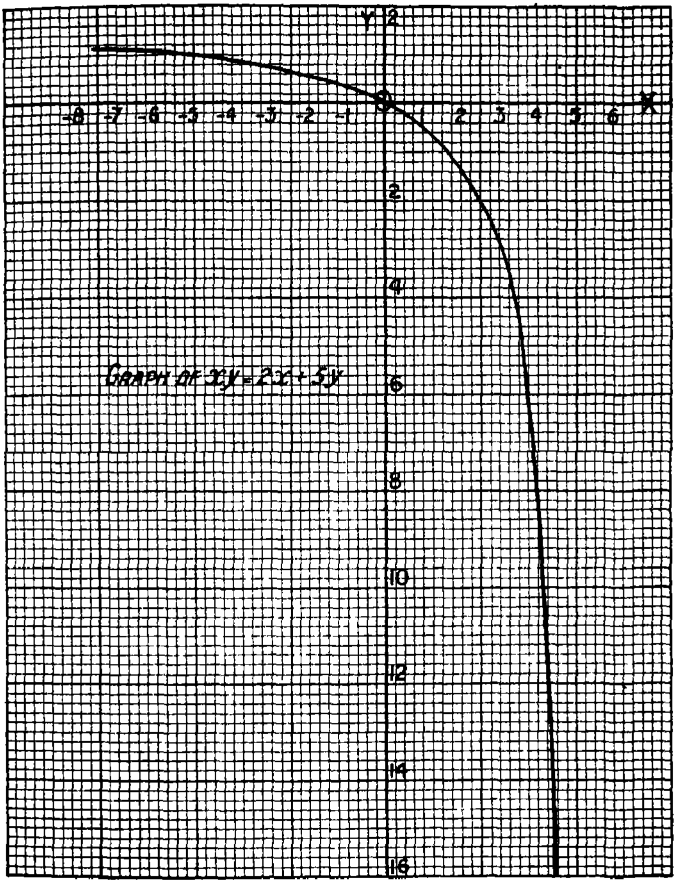
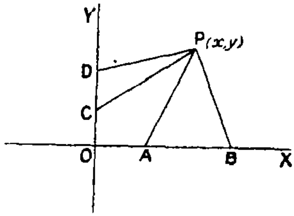


DIAGRAM 9

The graph table shows how rapid is the increase of x as y approaches 2, and of y as x approaches 5.

x	−45	−20	−11·67	−7·5	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	4·5	4·8	4·9
y	1·8	1·6	1·4	1·2	1·17	1·09	1	·89	·75	·57	·33	0	−·5	−1·33	−3	−8	−18	−48	−98

EXAMPLE 8.—*A length AB of 4 units is taken on OX, and a length CD of 2 units on OY. A variable point P moves so that the sum of triangles PAB and PCD is constant. Find the locus of P.*



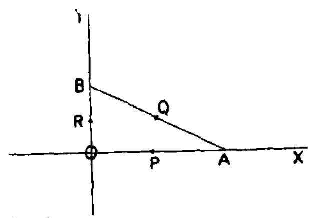
Let $P \equiv (x, y)$.

Then $2\Delta PCD = CD \times x = 2x$, and $2\Delta PAB = AB \times y = 4y$.

Now $\Delta PCD + \Delta PAB = c$ (a constant),

$$\therefore 2x + 4y = 2(\Delta PCD + \Delta PAD)$$
$$= 2c,$$
$$\therefore x + 2y = c.$$

The locus of P is the straight line which is the graph of this equation.



EXAMPLE 9.—*OAB is a triangle such that $A \equiv (4, 0)$ and $B \equiv (0, 2)$. Prove that its medians are concurrent.*

Let P be the mid-point of OA, Q the mid-point of AB, and R the mid-point of OB.

Then $P \equiv (2, 0)$, $Q \equiv (2, 1)$, and $R \equiv (0, 1)$.

Equation to PB is $\qquad \dfrac{x}{2} + \dfrac{y}{2} = 1 \quad . \quad . \quad . \quad . \quad (1)$

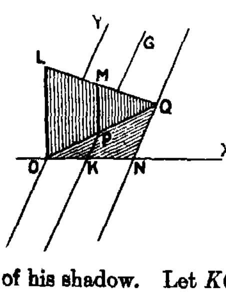
Equation to AR is $\qquad \dfrac{x}{4} + \dfrac{y}{1} = 1 \quad . \quad . \quad . \quad . \quad (2).$

Equation to OQ is

$$\frac{x}{2} = \frac{y}{1} \text{ (see Art. VI. Ex. 3)} \quad . \quad . \quad . \quad (3).$$

If equations (1) and (2) be solved it will be found that BP meets AR at the point $(\frac{4}{3}, \frac{2}{3})$.

By equation (3) this point lies on OQ if $\dfrac{4}{6} = \dfrac{2}{3}$, which is true.



Hence the three lines all pass through the point $(\frac{4}{3}, \frac{2}{3})$.

EXAMPLE 10.—*A man is passing down a straight street at night. Show that the end of his shadow as cast by a lamp follows a path parallel to the street.*

Take the line of the street through O, the foot of the lamp-post OL, as axis of y, and the cross line through the same spot as axis of x. Let MP represent the man at some moment. Then Q will be the end of his shadow. Let KG be his path.

Let $OL = b$, $PM = a$, and $Q \equiv (x, y)$.
Draw QN perpendicular to OX.
Then by similar triangles we have

$$\frac{OL}{PM} = \frac{OQ}{PQ} = \frac{ON}{KN},$$

$$\therefore \frac{b}{a} = \frac{x}{x - OK},$$

$$\therefore bx - b \cdot OK = ax,$$

which gives
$$x = \frac{b \cdot OK}{b - a},$$
$$= \text{constant}.$$

Now the graph of this equation is a straight line parallel to OY (Art. V. Cor. 2). It is the locus of Q.

The end of the shadow therefore follows a path parallel to the street.

RÉSUMÉ

1. The graph of the equation $lx + my + n = 0$ is a straight line parallel to LM, where $L \equiv (o, l)$ and $M \equiv (m, o)$.

2. The graph of the equation $lx + my = 0$ is a straight line passing through O.

3. The graphs of the equations $lx + n = 0$ and $my + n = 0$ are straight lines parallel to YY' and XX' respectively.

4. The equation to a straight line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

5. To find where two straight lines intersect, solve their equations.

EXAMPLES

1. Draw the graph of the equation $2x - 5y = 4$.

2. In the equation $3x + 4y = 12$ find the value of x when $y = 0$ and the value of y when $x = 0$.
What points therefore lie on the graph?
Hence draw the line.

3. Write down the equations of the lines which cross the axes at the following points:
(5, 0) and (0, 3); (−4, 0) and (0, 2); (−6, 0) and (0, −9); $(x_1, 0)$ and $(0, -y_1)$.

4. By finding where they cross the axes draw the straight lines which are the graphs of the following equations:

$$\frac{x}{2}+\frac{y}{3}=1;\quad -\frac{x}{2}+\frac{y}{3}=1;\quad -\frac{x}{2}-\frac{y}{3}=1;\quad \frac{x}{2}-\frac{y}{3}=1.$$

If the points of intersection with the axes are A, B, A', and B', what is the area of the rhombus $ABA'B'$?

5. Throw the following equations into intercept form:

$$4x+2y=9;\quad 5x-6y+3=0;\quad 2x=3y+5;\quad y=2x-4.$$

6. Find A and B, the points where the straight line $12x+5y=60$ crosses the axes.

What is the length of AB?

What is the area of $\Delta\, OAB$?

Deduce the distance of O from AB correct to two decimal places.

7. Make the construction of Article III. for the equation

$$2x+5y=16.$$

What is the area of the quadrilateral $OMPL$?

What is the area of $\Delta\, OML$?

Deduce the height of $\Delta\, PML$ correct to the first place of decimals.

8. Make the construction of Art. III. for the equation $3x+4y=20$.

Find the height of $\Delta\, PML$ and so draw the graph of the equation.

By finding from its equation two points on the graph, show that you have drawn the true straight line.

9. In the case of the equation $3x+4y=4$ show that the area of triangle OML is greater than that of the quadrilateral $OMPL$. Draw the graph in this case and compare it with that of the previous example.

10. How must OM be drawn for the equation $2x-3y=8$?

11. Find in the ordinary way the points A and B where the straight line $4x-5y+32=0$ crosses the axes.

Plot them and also the points M and L obtained as in Article III

Prove that $\dfrac{OM}{OA}=\dfrac{OL}{OB}$, thereby demonstrating that the lines AB and LM are parallel.

12. A straight line cuts the axes at A and B.

Take any point $P\equiv(x, y)$ lying on it.

If $A\equiv(a, o)$ and $B\equiv(o, b)$, use the fact that

$$\Delta\, OPB+\Delta\, OPA=\Delta\, OAB$$

to obtain the equation to AB in the form $\dfrac{x}{a}+\dfrac{y}{b}=1$.

13. A point moves so that its distance from the x-axis is one-third of its distance from the y-axis.

Find and draw its locus.

14. A variable point moves so that the sum of its distances from

two fixed straight lines at right angles to one another is always 5 units. Find its locus.

15. Perpendiculars PN and PM are drawn from a variable point P to the axes. If the length of the rectangle $ONPM$ exceeds the breadth by 2 units, find and draw the locus of P.

16. The perimeter of a rectangle is 18 units. If two of its adjacent sides are fixed in position find the locus of the vertex opposite their intersection and draw it.

17. The straight line $2x+5y=10$ cuts the axes at A and B. Find the co-ordinates of C the mid-point of AB. Find also the length of OC and its equation.

18. A straight line cuts the axes at A and B so that $OA=a$ and $OB=b$. Find the co-ordinates of C the mid-point of AB. Find also the equation and length of OC.

19. From a variable point P perpendiculars PN and PM are drawn to XX' and YY' respectively, so that the area of the rectangle $ONPM$ is always 20 sq. units. If $P\equiv(x_1, y_1)$ write down

(i.) the co-ordinates of N,
(ii.) the co-ordinates of M,
(iii.) the co-ordinates of Q the mid-point of MN.

Find the equation to the locus of Q and draw its graph.

20. Find the equation to the lines joining the points (3, 1) and (8, 4); (−2, 3) and (4, −5); (0, 6) and (−3, −2); (0, 0) and (−3, 2·5).

21. Perpendiculars PN and PM are drawn from the point (6, 4) to XX' and YY' respectively. N is joined to L the mid-point of PM. Find the equation to NL. Find also where NL cuts OY.

22. Lengths OA and OB are taken on the axes of x and y respectively. The rectangle $OACB$ is completed. If $OA=a$ and $OB=b$, find the equation to the line joining B to D the mid-point of AC. Find also where this line crosses OX.
(Use intercept form.)

23. In last example if BD cuts OX at E, find the equation to EC.

24. Find the intersection of the lines

(i.) $3x-5y+1=0$ and $4x-3y=6$,
(ii.) $3x-4y=22$ and $x-2y+6=0$,
(iii.) $5x+4y+35=0$ and $7x-6y+20=0$.

Verify the first result graphically.

25. Find the equation to the line joining the origin to the point of intersection of the lines $2x-3y=13$ and $4x+9y=11$.

26. A buttress reaches a height of 25 ft. on a vertical wall, and has a ground length of 10 ft. How far from the wall will a man 6 ft. high just emerge from behind the buttress?

27. A straight line is drawn parallel to the x-axis at a distance of

5 units above it. If P is any point on this line, find the locus of the mid-point of OP and draw it.

28. A straight line is drawn cutting the axes at N and M. If $\frac{ON}{OM}=$ constant, find the locus of the mid-point of MN.

29. A and B are two fixed points 4 units apart. P is a variable point such that $PA^2-PB^2=16$. Find and draw the locus of P.

30. $A\equiv(a, o)$ and $B\equiv(o, b)$. A variable point P moves so that $PA=PB$. Prove that the locus of P is a straight line.

31. The diagonal of a rectangle, two of whose sides are along OX and OY, cuts the axes at P and Q, and always passes through *the fixed point (5, 3).*

Prove that $5OQ+3OP=OP\cdot OQ$,

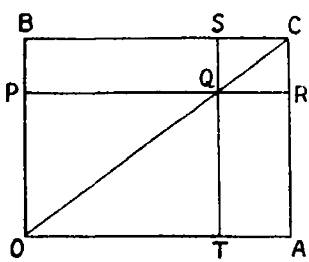
(i.) by taking the equation to PQ in intercept form,

(ii.) geometrically, by the help of areas.

32. Two boys are playing on two adjacent walks round a rectangular garden lawn, in such a way that they always keep between them an upright pole stuck in the ground. If the pole is c feet distant from each walk, find the relation which connects the distances of the boys from the corner of the lawn.

33. On a given straight line any point P is taken. A perpendicular PN is drawn to XX'. Prove that the locus of its mid-point is a straight line.

34. Points A and B are taken on the axes so that $A\equiv(a, o)$ and $B\equiv(o, b)$. The rectangle $OACB$ is completed. A point Q is taken on the diagonal OC. (See figure.)



(i.) Write down the equation to OC.

(ii.) If $Q\equiv(x', y')$ express QR and QS in terms of the co-ordinates of A, B, and Q.

(iii.) Hence prove $BPQS=TARQ$.

35. The point $P\equiv(h, k)$ is joined to the origin and to $A\equiv(a, o)$. Find the co-ordinates of Q and R the mid-points of OP and AP respectively. Use the results to prove QR parallel to XX' and equal to $\frac{OA}{2}$.

36. Two poles of height 4 ft. and 20 ft. respectively are set in the ground. Their ends are joined crosswise by strings. If one of the poles be moved about, prove that the intersection of the strings lies on a horizontal plane.

37. A man is walking along a straight road on a sunny day. *Prove that the end of his shadow pursues a path parallel to the road.*

38. A man is passing down a straight street by night. Prove that the end of his shadow, as cast by a certain lamp, describes a straight line parallel to his path.

39. A and B are two fixed points on XX', C and D two fixed points on YY'. P moves so that $\Delta PCD + \Delta PAB$ is constant. Prove that the locus of P is a straight line.

40. In last example find the locus of P if $\dfrac{\Delta PCD}{\Delta PAB}$ is constant.

41. The base AB of a triangle PAB is fixed, and its area is constant. Show that the locus of its median centre is a straight line.

42. $OACB$ is a square. A point P is taken on AB. Prove analytically that $PM+PN$ is constant where PM is perpendicular to OB and PN to OA.

43. A straight line cuts the axes at A and B. Any point P is taken on it. Show that the locus of Q, the mid-point of OP, is a straight line parallel to AB.

44. Prove analytically that the diagonals of a rectangle bisect each other, taking two adjacent sides as axes.

45. Points $A \equiv (1, 0)$, $B \equiv (0, 4)$, and $H \equiv (2, 3)$ are taken.

(i.) Find where AH and BH cut the axes again at P and Q. (Hint: Use intercept form.)

(ii.) Write down the equation to PQ.

46. From a point $C \equiv (a, b)$ perpendiculars CA and CB are drawn to the axes of x and y respectively. D is the mid-point of OA and E of BC.

(i.) Find the equations of BD and AE respectively.

(ii.) Show that BD and AE cut OC at points of trisection.

47. A straight line swings about the point $(3, 3)$, and cuts the axes at P and Q. Prove that $\dfrac{1}{OP}+\dfrac{1}{OQ}=\dfrac{1}{3}$. Find the equation to PQ when $\Delta OPQ = 24$ sq. units.

48. A square $ABCD$ is set with the corners A and B on OX and OY respectively, so that $OA = a$ and $OB = b$. If CD is turned away from O, find where OC and OD cut AB.

49. A straight line cuts the axes at A and B and is intersected by the bisector of $X\hat{O}Y$ at the point C. Find the lengths of AC and BC, and so prove that $\dfrac{AC}{CB}=\dfrac{OA}{OB}$.

50. Squares are set with one corner on OX and one on OY. Find the locus of their centres.

51. A straight line cuts the axes at A and B. A point P is found within ΔOAB such that $\Delta POA = \Delta POB = \Delta PAB$. What are the co-ordinates of P?

52. A straight line cuts the axes at A and B.

If $\Delta\, PAB = \frac{1}{3}\, \Delta\, OAB$, prove that the locus of P is a straight line.

53. $A \equiv (a, o)$, $B \equiv (o, b)$, $A' \equiv (ka, o)$ and $B' \equiv (o, kb)$.

Find where AB' and $A'B$ intersect.

If this point is joined to the origin prove that the join cuts AB at its mid-point. (Hint: Show that the co-ordinates of the mid-point satisfy the equation of the join.)

54. The ratio of the intercepts $(a : b)$ which a straight line makes on the axes is $5 : 4$. If it also passes through the intersection of the lines $2x + 3y = 5$ and $3x - 4y = 33$, find its equation.

55. A and B are the points $(1, 0)$ and $(0, 1)$. The square of which OA and OB are sides is completed. Find the equation to the line which passes through the intersection of the diagonals and the point $(3, 0)$.

56. From a variable point P perpendiculars PN and PM are drawn to the axes. The diagonal MN of the rectangle so formed always passes through the point $(2, 1)$. Find the locus of P, and draw a part of it.

57. CAB is a triangle right-angled at C. P is a variable point in its plane, such that $\Delta\, CAP = \Delta\, CBP$ in area.

Prove that the locus of P is the median through C of $\Delta\, CAB$.

Interpret the equation $lx + my = o$ in the light of this result.

58. Make use of the fact that *if a triangle be on a fixed base and has a constant area the locus of its vertex is a straight line*, to prove that

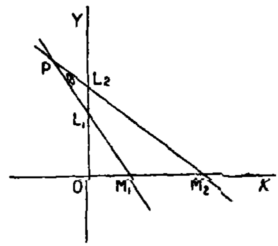
(i.) the graph of $lx + n = 0$ is a parallel to YY', and

(ii.) the graph of $my + n = 0$ is a parallel to XX'.

CHAPTER III

ANGLE BETWEEN TWO LINES : PARALLEL AND PERPENDICULAR LINES : GRADIENTS : THE EQUATION $y = mx + b$

I. *Find the tangent of the angle between the two lines whose equations are $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$.*



Let θ be the angle between the lines.

On OX mark off $OM_1 = m_1$ and $OM_2 = m_2$.

On OY mark off $OL_1 = l_1$ and $OL_2 = l_2$.

Then by Chapter II. Article III. we know that the line

$$l_1x + m_1y + n_1 = 0$$

is parallel to L_1M_1, and that the line $l_2x + m_2y + n_2 = 0$ is parallel to L_2M_2.

Hence the angle M_1PM_2 between the lines L_1M_1 and L_2M_2 will be θ.

$$\tan\theta = \tan(OM_1L_1 - OM_2L_2)$$

$$= \frac{\tan OM_1L_1 - \tan OM_2L_2}{1 + \tan OM_1L_1 \cdot \tan OM_2L_2}$$

$$= \frac{\dfrac{l_1}{m_1} - \dfrac{l_2}{m_2}}{1 + \dfrac{l_1}{m_1} \times \dfrac{l_2}{m_2}}$$

$$= \frac{l_1m_2 - l_2m_1}{l_1l_2 + m_1m_2}.$$

Corollary 1.—*The lines are parallel if* $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2}$.

If the lines are parallel, $\hat{\theta}=0$ since they are not inclined. Now $\tan 0=0$,

$$\therefore \frac{l_1m_2-l_2m_1}{l_1l_2+m_1m_2}=0.$$

$$\therefore l_1m_2-l_2m_1=0,$$

which gives
$$\frac{l_1}{l_2}=\frac{m_1}{m_2}.$$

Corollary 2.—The lines are perpendicular if $l_1l_2+m_1m_2=0$.

If the lines are at right angles, then $\theta=90°$.

But $\tan 90°=\infty$.

$$\therefore \frac{l_1m_2-l_2m_1}{l_1l_2+m_1m_2}=\infty.$$

$$\therefore l_1l_2+m_1m_2=0.$$

Remarks.—(i.) The reader will more readily apprehend the idea of an infinite ratio if he remembers that when a large number is divided by a very small one then the quotient is big. For example, if we divide 23 by ·000001, the result is 23,000,000. He will then easily appreciate why the divisor, or denominator, in Corollary 2 must be infinitely small to give an infinitely large quotient.

(ii.) The following scheme is helpful in remembering the results of this article. Place the coefficients of x and y in one equation above those of the same variables in the other equation thus: $\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$. Then we notice that $\tan\theta$ is the difference of the cross products over the sum of the vertical ones. Corollary 1 expresses the equality of the ratios of the vertical letters. Corollary 2 states that the sum of the vertical products is zero.

EXAMPLE 1.—*Find the angle between the lines $2x-5y=1$ and $6x-4y=3$.*

We have
$$\tan\theta=\frac{l_1m_2-l_2m_1}{l_1l_2+m_1m_2}$$
$$=\frac{-30+8}{12+20}$$
$$=-·6875$$
$$\therefore \theta=145°\ 30'.$$

The acute angle between the lines is 34° 30′.

EXAMPLE 2.—*Prove that the lines* $2x+5y=6$ *and* $6x+15y+7=0$ are parallel.

$$\frac{l_1}{l_2}=\frac{2}{6}=\frac{1}{3} \text{ and } \frac{m_1}{m_2}=\frac{5}{15}=\frac{1}{3},$$

$$\therefore \frac{l_1}{l_2}=\frac{m_1}{m_2}.$$

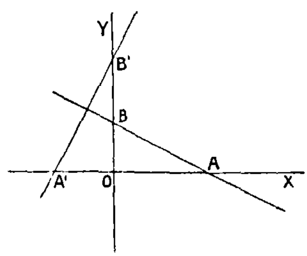
Hence by Corollary 1 the lines are parallel.

EXAMPLE 3.—*Prove that the lines* $2x-5y=2$ *and* $10x+4y+3=0$ *are perpendicular.*

$$l_1l_2+m_1m_2=(2\times 10)+(-5\times 4)=0.$$

The lines are therefore perpendicular by Corollary 2.

EXAMPLE 4.—*Two straight lines at right angles to one another are drawn cutting the axes at A, B and A′, B′ respectively. Prove that the rectangles OA·OA′* and *OB·OB are equal.*



If $OA=a$, $OB=b$,
$OA'=a'$ and $OB'=b'$,
then the equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1$$

and that of $A'B'$ is

$$\frac{x}{a'}+\frac{y}{b'}=1.$$

Since these two lines are perpendicular,

$$\therefore \left(\frac{1}{a}\times\frac{1}{a'}\right)+\left(\frac{1}{b}\times\frac{1}{b'}\right)=0$$

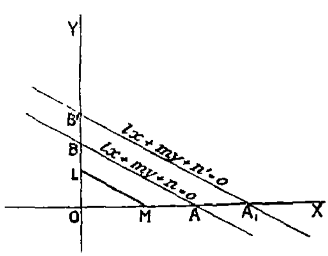
$$\therefore bb'+aa'=0$$

$$\therefore bb'=-aa'$$

$$\therefore OB{\cdot}OB'=OA{\cdot}OA',$$

the sign being immaterial as regards the areas of the rectangles.

II. *Two straight lines are parallel, whose equations differ only in the absolute terms.*



This is so important that we are giving the proof here, though it has already been given in Chapter II., and can be deduced from Corollary 1 of last article.

Let the equations to the lines be $lx+my+n=0$ and $lx+my+n'=0$.

On OX take M, and on OY take L, so that $OM=m$ and $OL=l$.

Then we know from Chapter II. that the graphs of the given equations are straight lines parallel to LM.

They are therefore parallel to each other

EXAMPLE.—*Find the equation to a straight line passing through the point* $(3, -2)$ *and parallel to* $5x+4y-2=0$.

The required equation will differ from $5x+4y-2=0$ only in the absolute term.

It will therefore have the form $5x+4y+n=0$.

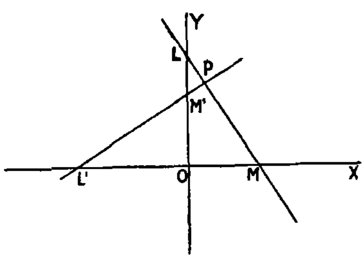
Now the point $(3, -2)$ is on it,

$$\therefore 15-8+n=0. \quad \therefore n=-7.$$

Hence the equation is

$$5x+4y-7=0.$$

III. *Two straight lines are perpendicular in whose equations the coefficients of x and y are interchanged and the sign of one coefficient reversed.*



This is an immediate deduction from Corollary 2 of Article I., but is so important that we shall give an independent proof.

Let the equations to the lines be

$$lx+my+n=0 \text{ and } mx-ly+n'=0.$$

In the figure $OM=OM'=m$, $OL=l$ and $OL'=-l$.

Then we know that $lx+my+n=0$ is parallel to LM, and $mx-ly+n'=0$ is parallel to $L'M'$. Hence we have only to show that LM is perpendicular to $L'M'$ to prove our theorem.

Plainly triangles $OL'M'$ and OLM are congruent,

$$\therefore O\hat{L}'M'=O\hat{L}M.$$

But $$O\hat{M}'L'=P\hat{M}'L.$$

Hence Δ s $OL'M'$ and $PM'L$ are equiangular,

$$\therefore L\hat{P}M'=L'\hat{O}M'=90°.$$

The lines are therefore perpendicular.

EXAMPLE 1.—*Find the equation to a straight line through the point* (3, 1) *perpendicular to* $3x-4y+2=0$.

The form of the required equation will be $4x+3y+n=0$, for the coefficients of x and y in this equation are those of y and x in the given one, with the sign of the coefficient of y reversed.

Now (3, 1) is a point on $4x+3y+n=0$,

$$\therefore\ 12+3+n=0,$$
$$\therefore\ n=-15.$$

Hence the equation we seek is $4x+3y-15=0$.

EXAMPLE 2.—*Apply Corollary 2 Article I. to prove that the lines* $3x-4y+2=0$ *and* $4x+3y-15=0$ *are perpendicular.*

$$\begin{aligned} l_1l_2+m_1m_2 &= (3\times 4)+(-4\times 3)\\ &= 12-12\\ &= 0. \end{aligned}$$

The lines are therefore perpendicular.

IV. Every reader will be familiar with the expression "A gradient of 1 in 100." This means that if an engine is moving up an incline, then for every 100 yards it proceeds, its distance

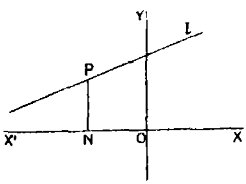


FIG 1

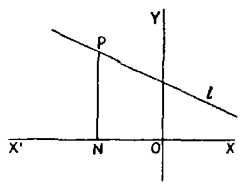


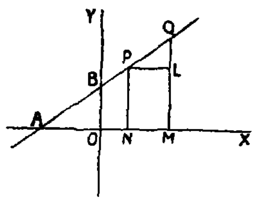
FIG. 2

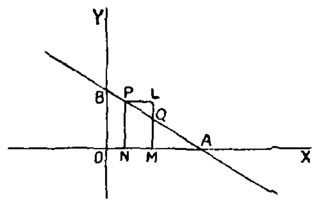
above the level will increase by 1 yard. The use of the term gradient is slightly different in analytical geometry.

Imagine a point N to start from the negative end of the x-axis and to proceed towards the positive end, while the head of an ordinate NP travels along a straight line l. Then the distance of P from the line XX' will increase (Fig. 1) or decrease (Fig. 2) by a fixed amount for every unit of distance traversed by N, as we shall hereafter prove. The ratio of the rise or fall of P to the corresponding distance travelled by N is called the gradient of the line. Such a line as in Fig. 1 where P is rising is called an ascending line, while one like that of Fig. 2, where P is falling, is called a descending line.

V. *The gradient of a given straight line is constant for that line.*

Let a straight line cut the axes at A and B.





Let the point P proceed from P to Q, while N travels from N to M.

Draw PL parallel to XX'.

Then by definition the gradient of AB is the ratio $\frac{LQ}{NM}$, which is equal to $\frac{LQ}{PL}$.

Now triangles PLQ and AOB are evidently similar,

$$\begin{aligned} \therefore \text{gradient} &= \frac{LQ}{PL} \\ &= \frac{OB}{AO} \\ &= \text{constant.} \end{aligned}$$

We note that in a descending line LQ is turned downwards. Hence the gradient of a descending line is negative.

Corollary 1.—*The gradient of a line is measured by the ratio of the difference of the ordinates to the difference of the abscissae of any two points on it.*

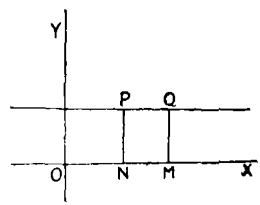
$$\begin{aligned} \text{We have the gradient} &= \frac{LQ}{PL} \\ &= \frac{MQ - NP}{OM - ON} \\ &= \frac{\text{difference of the ordinates}}{\text{difference of the abscissae}}. \end{aligned}$$

This also shows that the gradient of a descending line is

negative for then MQ is less than NP, so that $MQ-NP$ is negative. The construction being such that $OM-ON$ is always positive it follows that the gradient of a descending line is negative.

Corollary 2.—*The gradient of the line joining the points* (x_1, y_1) *and* (x_2, y_2) *is* $\frac{y_2-y_1}{x_2-x_1}$, *since this expression gives the ratio* $\frac{\text{difference of ordinates}}{\text{difference of abscissae}}$.

If the line passes through the origin, then $x_1=y_1=0$, and the gradient will be $\frac{y_2}{x_2}$.



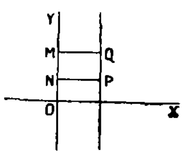
Corollary 3.—*If a line is parallel to the x-axis its gradient is zero.*

Let P and Q be two points on the line.

Then since PQ is parallel to XX', therefore $NP=MQ$.

Hence
$$y_2=y_1,$$
$$\therefore \text{gradient}=\frac{y_2-y_1}{x_2-x_1}=\frac{0}{x_2-x_1}=0.$$

We have here the idea of an absolutely level road.



Corollary 4.—*If a line is parallel to the y-axis its gradient is infinite.*

In this case $x_2=x_1$ for $NP=MQ$,
$$\therefore \text{gradient}=\frac{y_2-y_1}{x_2-x_1}=\frac{y_2-y_1}{0}=\infty.$$

We have now the impression of great steepness, of "sheer descent" as from a cliff.

Example 1.—*What is the gradient of the line joining the points* (7, 9) *and* (2, 6)?

$$\text{Gradient}=\frac{y_2-y_1}{x_2-x_1}=\frac{9-6}{7-2}=\frac{3}{5}.$$

Example 2.—*Show by a diagram that the line joining the points* $(-2, 1{\cdot}7)$ *and* $(-{\cdot}5, 1)$ *is a descending one, and calculate its gradient.*

(i.) In Diagram 1 we notice that LQ is drawn downwards, so that the gradient $\frac{LQ}{PL}$ is negative as happens in a descending line.

(ii.) $$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1{\cdot}7 - 1}{-2 + {\cdot}5} = -\frac{7}{15}.$$

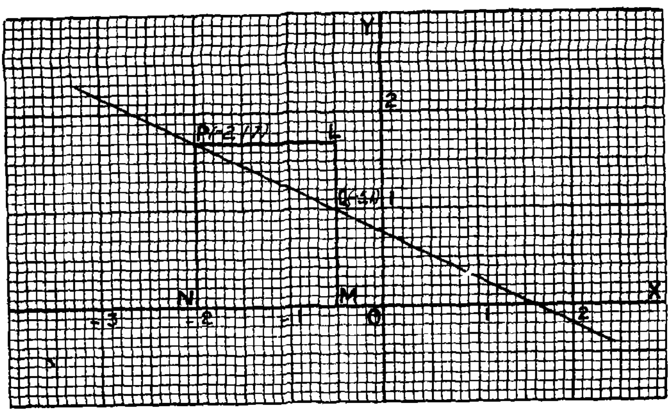


Diagram 1.

Example 3.—*Draw an ascending straight line of gradient* $\frac{3}{2}$ *passing through the point* $(2, 2)$.

Let $P \equiv (2, 2)$. In this example LQ is drawn upwards. (Diagram 2.) To draw the line we count 2 units from P to L going from left to right, and then 3 units upwards to Q. Join PQ.

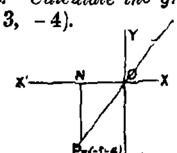
Example 4.—*Draw a line of gradient* $-\frac{3}{5}$ *through the point* $(-1, -2)$.

The gradient being negative, the line is a descending one, so that LQ must be drawn downwards. (Diagram 3.)

Let $$P \equiv (-1, -2).$$

From P count 5 units to L going from left to right and then 3 units downwards to Q. Join PQ.

Example 5.—*Calculate the gradient of the line joining the origin to the point* $(-3, -4)$.



$$\text{Gradient} = \frac{y_1}{x_1} = \frac{-4}{-3} = \frac{4}{3}.$$

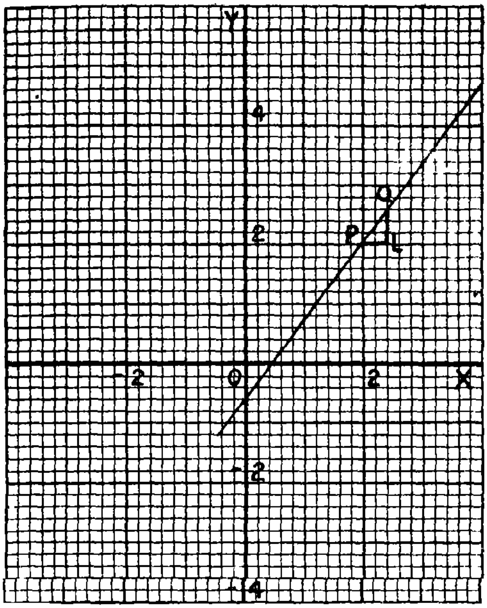


DIAGRAM 2.

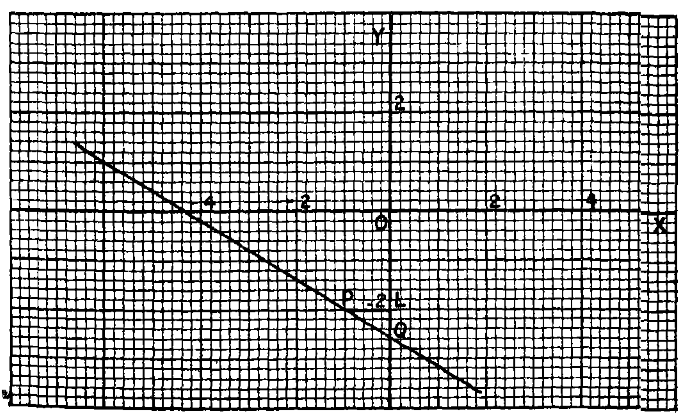
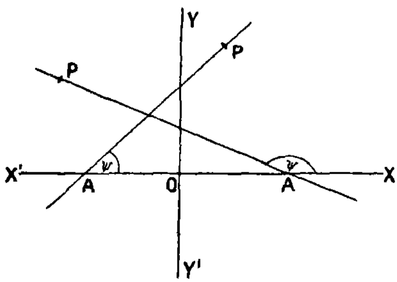


DIAGRAM 3.

VI. *The slope of a straight line.*

Definition.—If a point P lying in the first or second quadrants be taken on a straight line which cuts the x-axis at A, then $X\hat{A}P$ is called the slope of the line (see figure subjoined).



The angle XAP is usually called ψ (Greek letter psi).

It may be acute or obtuse.

The angle $X'AP$ must not be taken as the slope, of which it is the supplement.

Corollary.—tan ψ = gradient of the line.

On referring back to the figure of Article V. we see that

$$\begin{aligned}\tan\psi &= \tan XAP\\ &= \frac{OB}{AO}\\ &= \text{gradient of the line.}\end{aligned}$$

The fact that tan ψ measures the gradient of the line is very important. It enables us to find the slope of a line when its gradient is known.

EXAMPLE 1.—*Find the slope of the line joining the points* (1, 2) *and* (4, 7) *by measurement and by calculation.*

On plotting the points and joining them we find by measurement that $\psi = X\hat{A}P$ is rather more than 59°. (Diagram 4.)

Again the gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-7}{1-4} = \dfrac{5}{3}.$

$$\therefore \tan\psi = 1{\cdot}6667 \text{ (nearly).}$$

Hence from mathematical tables we obtain

$$\psi = 59^\circ\ 2'.$$

EXAMPLE 2.—*A line passes through the point* (0, 2) *and has a slope of* 122°. *Draw it and find its gradient from the diagram and from mathematical tables.*

Diagram 4 shows us that the line passes through the point (−2, 5·2).

Hence the gradient $=\frac{LB}{PL}=\frac{-1\cdot6}{1}=-1\cdot6.$

From the tables we have the gradient $=\tan 122° = -1\cdot6003.$

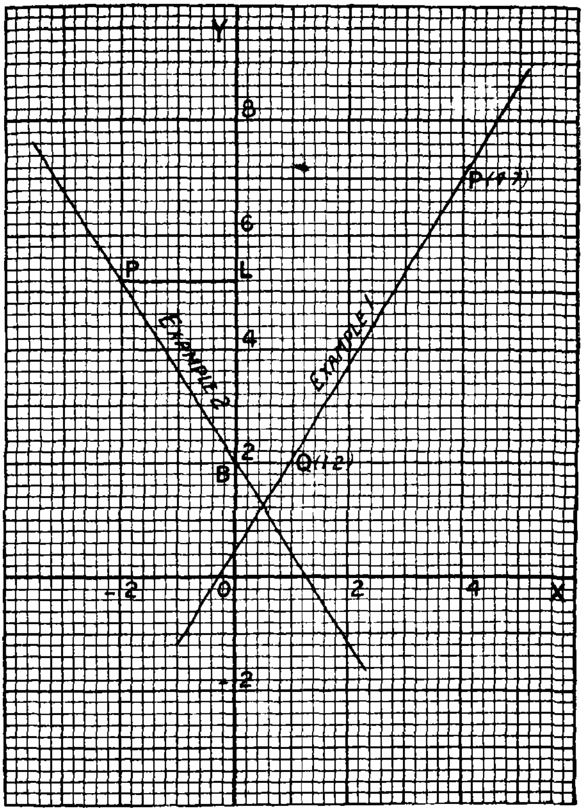


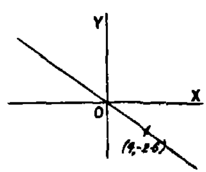
DIAGRAM 4.

EXAMPLE 3.—*Calculate the slope of the line joining the origin to the point* (4, − 2·6).

$$\text{Gradient } = \frac{y_1}{x_1} = -\frac{2\cdot6}{4}.$$

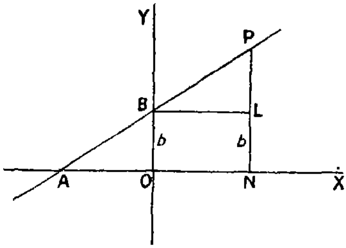
$$\therefore \tan\psi = -\cdot65.$$

$$\therefore \psi = 146° \ 58' \ 30''.$$



These examples will bring out the fact that a positive gradient means an acute slope, while a negative gradient means an obtuse slope.

VII. ***Find the equation to a straight line of gradient m which intercepts a length of b units on the y-axis.***



OB is the intercept of the line on the *y-axis*, and is therefore b units long.

Let $P \equiv (x, y)$ be any other point on the line.

Draw PN perpendicular to, and BL parallel to, the *x-axis*.

We have the gradient $= \dfrac{LP}{BL} = m.$

$$\therefore \frac{NP - NL}{ON} = m.$$

$$\therefore \frac{y - b}{x} = m.$$

Whence $$y = mx + b.$$

In this equation there are two constants, m and b, and two variables, x and y.

If the straight line passes through the origin, then b is of zero length, and the equation reduces to $y = mx$.

Corollary 1.—*If ψ is the slope of the line, then tan ψ = gradient $= m$.*

We could therefore write the equation just found thus :

$$y = x \tan \psi + b.$$

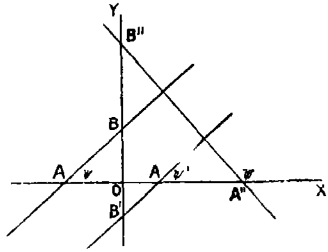
Since $B \equiv (o, b)$ we might say that we have found the equation to a straight line passing through the point (o, b), and having a given slope ψ.

The " gradient " form is, however, of much greater importance than the " slope " form of the equation.

Corollary 2.—*Parallel straight lines have equal gradients.*

If the lines AB and $A'B'$ in the figure are parallel, then evidently $\psi = \psi'$.

$$\therefore \tan \psi = \tan \psi'.$$



Hence the gradients are equal

It follows that if the equations to AB and $A'B'$ are

$$y = mx + b$$

and $\qquad y = m'x + b'$,

then $\qquad m = m'$

Corollary 3.—*If two lines are perpendicular, then the product of their gradients is* -1

Let $A''B''$ be perpendicular to AB

Then
$$\tan\psi'' = -\tan AA''B''$$
$$= -\cot\psi$$
$$= -\frac{1}{\tan\psi}$$
$$\tan\psi \tan\psi'' = -1$$

That is, the product of the gradients is -1

If the equation to $A''B''$ is $y = m''x + b''$, that of AB being $y = mx + b$, then $mm'' = -1$. In other words $m'' = -\dfrac{1}{m}$.

Both these corollaries could easily be deduced by the help of Article I, but their importance warrants an independent proof.

EXAMPLE 1.—*Write down the equation to a straight line of gradient* $\frac{2}{3}$ *which makes a negative intercept of* 4 *units on the y axis.*

We have $\qquad m = \frac{2}{3}$ and $b = -4$.

The equation is therefore, by substitution in $y = mx + b$ of these values,

$$y = \tfrac{2}{3}x - 4,$$

which gives $\qquad 2x - 3y = 12.$

EXAMPLE 2.—*What is the slope of the line* $4x - 5y + 6 = 0$?

We must first find the gradient of the line, and to do so it is necessary to write the equation in the form $y = mx + b$.

We have
$$5y = 4x + 6$$
$$\therefore\ y = \tfrac{4}{5}x + \tfrac{6}{5}$$

The gradient is therefore $\dfrac{4}{5}$.

$$\therefore \tan\psi = \cdot 8$$
$$\therefore \psi = 38^\circ\ 40' \text{ nearly}$$

The line cuts yy' at the point (0, 1·2) for by the last form of the equation $b = \frac{6}{5}$.

EXAMPLE 3.—*Find the gradient of the line $lx + my + n = 0$.*

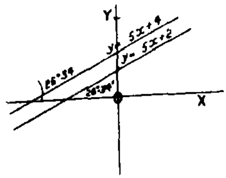
We first of all write

$$my = -lx - n$$

$$\therefore y = -\frac{l}{m}x - \frac{n}{m}.$$

We can now see that the gradient is $-\frac{l}{m}$.

The line makes an intercept $-\frac{n}{m}$ on the y-axis.



EXAMPLE 4.—*Show that the straight lines $y = \cdot 5x + 2$ and $y = \cdot 5x + 4$ are parallel.*

The gradient of each is ·5, hence they are parallel.

To find their slope we have $\tan\psi = \cdot 5$ $\therefore \psi = 26° 34'$.

EXAMPLE 5. *Show that the straight lines $4x - 5y + 3 = 0$ and $8x - 10y + 9 = 0$ are parallel.*

The gradient of the first is $\frac{4}{5}$.

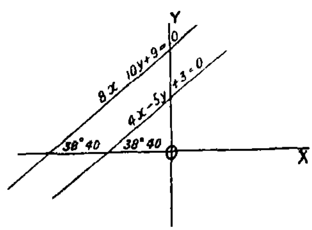
The gradient of the second is $\frac{8}{10}$ or $\frac{4}{5}$.

Thus the two lines are parallel since they have equal gradients.

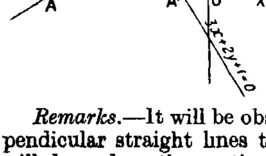
Their slope is given by

$$\tan\psi = \cdot 8.$$

$$\therefore \psi = 38° 40'.$$



EXAMPLE 6.—*Prove that the lines $2x - 3y + 4 = 0$ and $3x + 2y + 1 = 0$ are perpendicular and verify by a figure.*



(i.) The gradients of the lines are $\frac{2}{3}$ and $-\frac{3}{2}$.

Their product is therefore -1.

Hence the lines are perpendicular.

(ii.) Since $\tan PA'X = -\frac{3}{2}$

$$\therefore P\hat{A}'X = 123° 42'.$$

And since $\tan PAX = \frac{2}{3}$

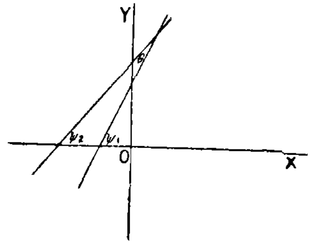
$$\therefore P\hat{A}X = 33° 42'.$$

$$\therefore A\hat{P}A' = 90°.$$

Remarks.—It will be observed that the tests for parallel and perpendicular straight lines take several forms. Which is to be used will depend on the particular problem. The fact that the product of the gradients of perpendicular lines is -1 is very important.

VIII. ***Find the tangent of the angle between two lines whose gradients are m_1 and m_2.***

Let θ be the angle between the lines and let ψ_1 and ψ_2 be their slopes.



Then

$$\tan \psi_1 = m_1 \text{ and } \tan \psi_2 = m_2$$

Now $\tan \theta = \tan (\psi_1 - \psi_2)$

$$= \frac{\tan \psi_1 - \tan \psi_2}{1 + \tan \psi_1 \tan \psi_2},$$

$$\therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Corollary 1.—If the lines are parallel, $\tan \theta = \tan 0° = 0$,

$$\therefore m_1 - m_2 = 0. \quad \therefore m_1 = m_2.$$

Corollary 2. If the lines are perpendicular, then

$$\tan \theta = \tan 90° = \infty,$$

$$\therefore 1 + m_1 m_2 = 0,$$

$$\therefore m_1 m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}.$$

Both these corollaries have already been proved.

EXAMPLE.—*Find the acute angle between the lines whose gradients are $\frac{4}{5}$ and $\frac{7}{6}$.*

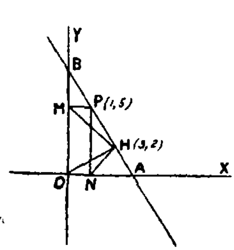
$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{4}{5} - \frac{7}{6}}{1 + \frac{4}{5} \cdot \frac{7}{6}} = -\cdot 1897 \text{ (nearly)},$$

$$\therefore \theta = 169° \ 16'.$$

The acute angle is $10° \ 44'$.

IX. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*A line is drawn through the point $H \equiv (3, 2)$ so as to be perpendicular to OH. Find its equation and show that the point $P \equiv (1, 5)$ lies on it.*



The equation to OH is

$$\frac{x}{3} = \frac{y}{2},$$

$\therefore 2x - 3y = 0.$ (Chap. II. Art. VI.)

Hence the equation to AB, which is perpendicular to OH, is of the form

$$3x + 2y + n = 0. \quad \text{(Art. III.)}$$

But $H \equiv (3, 2)$ is on the line,

$$\therefore 9+4+n=0,$$
$$\therefore n=-13.$$

The equation to AB is therefore $3x+2y-13=0$.
Now when $x=1$, $y=5$, so that the point $P \equiv (1, 5)$ lies on AB.

EXAMPLE 2.—*In Example* 1 *draw PN and PM perpendicular to the axes. Prove HM at right angles to HN.*

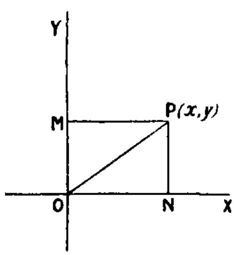
We have $M \equiv (0, 5)$, $N \equiv (1, 0)$, and $H \equiv (3, 2)$.

Gradient of $MH = \frac{5-2}{-3} = -1$. (Art. V. Cor. 2.)

Gradient of $NH = \frac{-2}{1-3} = 1$.

Product of gradients of MH and $NH = -1$.
The lines are therefore perpendicular (Art. VII. Cor. 3).

EXAMPLE 3.—*Three men start from the same place at the same time. One travels due east at* 4 *m.p.h., the second travels due north at* 3 *m.p.h., while the third keeps due north of the first and due east of the second. Find the direction in which the third man travels and his speed.*



(i.) Take the starting-place as origin, the East-West line as axis of x and the North-South line as axis of y.

Let the third man be at $P \equiv (x, y)$.

Then the first man is at N and the second at M.

At 4 m.p.h. the time taken to travel the distance ON is $\frac{ON}{4}$ hr.

At 3 m.p.h. the time taken to travel OM is $\frac{OM}{3}$ hr. But the first man goes from O to N while the second goes from O to M,

$$\therefore \frac{ON}{4} = \frac{OM}{3},$$

$$\therefore \frac{x}{4} = \frac{y}{3},$$

$$\therefore y = \tfrac{3}{4}x.$$

The slope of this line is given by $\tan \psi = \frac{3}{4} = \cdot 75$,

$$\therefore \psi = 36° \ 52'.$$

The third man travels along a straight road running in the direction 36° 52′ N.E.

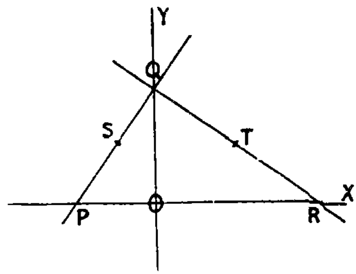
(ii.) If P is the position of the third man at the end of an hour then $ON=4$ and $PN=3$,

$$\therefore OP^2=16+9=25,$$

$$\therefore OP=5.$$

The speed of the third man is 5 m.p.h.

EXAMPLE 4.—*A system of parallel lines is drawn cutting the axes. Find the locus of the mid-points of the parts of them intercepted between the axes.*



Since the lines are parallel they have the same gradient k.

Let one of them cut the axes at P and Q.

Let $S\equiv(x', y')$ be the mid-point of PQ.

Let $Q\equiv(o, q)$.

Then the equation to PQ is $y=kx+q$ (Art. VII.).

Hence by the usual method $P\equiv\left(-\frac{q}{k}, 0\right)$

$$\therefore S\equiv\left(-\frac{q}{2k}, \frac{q}{2}\right), \quad \text{(Chap. I.)}$$

$$\therefore x'=-\frac{q}{2k} \text{ and } y'=\frac{q}{2}.$$

Eliminate q, which is a variable depending on the particular line of the system drawn, and must therefore not appear in the equation to the locus.

We have
$$\begin{aligned} y' &= \frac{q}{2} \\ &= -kx'. \end{aligned}$$

The locus of S is therefore the graph of the equation

$$y=-kx,$$

a straight line passing through the origin.

EXAMPLE 5.—*At every point Q of last example draw a perpendicular to PQ cutting XX' at R.*

Find the locus of T, the mid-point of RQ, and prove that it is perpendicular to the locus of S.

Since the gradient of PQ is k, that of RQ is $-\frac{1}{k}$ ($N.B.$ $mm'=-1$ gives $m=-\frac{1}{m'}$).

The equation to RQ is therefore

$$y = -\frac{1}{k}x + q,$$

$$\therefore\ R \equiv (kq,\ o);$$

and since $Q \equiv (o,\ q)$, $\quad \therefore\ T \equiv \left(\frac{kq}{2},\ \frac{q}{2}\right)$. (Chap. I.)

Let $\quad T \equiv (x',\ y'),$

$$\therefore\ x' = \frac{kq}{2} \text{ and } y' = \frac{q}{2}.$$

Eliminate q as before,

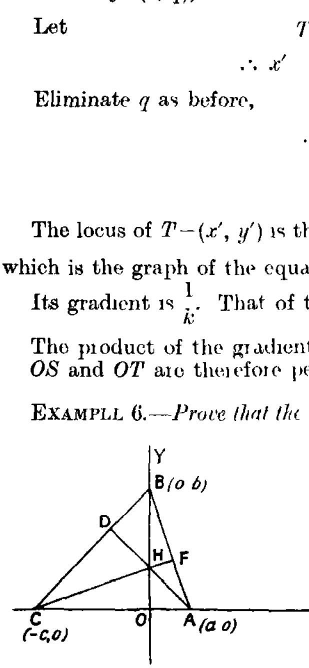
$$\therefore\ y' = \frac{q}{2} = \frac{x'}{k}.$$

The locus of $T \equiv (x',\ y')$ is the straight line through the origin, which is the graph of the equation $y = \frac{1}{k}x$.

Its gradient is $\frac{1}{k}$. That of the last example was $-k$.

The product of the gradients is -1.
OS and OT are therefore perpendicular lines.

EXAMPLE 6.—*Prove that the altitudes of a Δ ABC are concurrent.*



Take CA as axis of x and the perpendicular from B to CA as axis of y.

Let $A \equiv (a,\ o)$, $B \equiv (o,\ b)$, and $C \equiv (-c,\ o)$.

Let the altitude from C cut OY at $H \equiv (o,\ h)$.

The equation of AB is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and of CH is $\quad -\frac{x}{c} + \frac{y}{h} = 1.$

These lines are perpendicular,

$$\therefore\ \left(\frac{1}{a} \times -\frac{1}{c}\right) + \left(\frac{1}{b} \times \frac{1}{h}\right) = o, \quad \text{(Art. I.)}$$

$$\therefore\ -\frac{1}{ac} + \frac{1}{bh} = o,$$

whence $h = \frac{ac}{b}$, so that $H \equiv \left(o,\ \frac{ac}{b}\right)$.

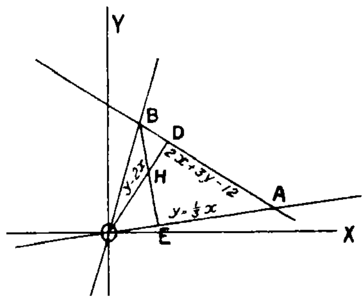
Similarly it can be shown that AD passes through $\left(o,\ \frac{ac}{b}\right)$.

The altitudes are therefore concurrent at H.

EXAMPLE 7.—*Find the orthocentre of the triangle whose sides are* $y=\frac{1}{3}x$, $y=2x$, *and* $2x+3y=12$.

The orthocentre of a triangle is the point at which its three altitudes meet.

We need therefore find the equations to two of these only, and solve them.



O is a vertex of the triangle, since two of its sides are seen by their equations to pass through the origin.

Let the remaining vertices be A and B as shown in the figure.

On solving the equations $y=2x$ and $2x+3y=12$, we find that $B\equiv(1{\cdot}5,\ 3)$.

Draw OD perpendicular to AB, and BE to OA.

Since the equation to AB is $2x+3y=12$, its gradient is $-\frac{2}{3}$. Thus the gradient of OD is $\frac{3}{2}$.

The equation to OD is therefore $y=\frac{3}{2}x$. . . . . (1)

Again the equation to OA is $y=\frac{1}{3}x$. Hence the gradient of BE is -3.

The equation to BE is therefore of the form $y=-3x+k$. Since $B\equiv(1{\cdot}5,\ 3)$ lies on this line,

$$\therefore\ 3=-4{\cdot}5+k,$$
$$\therefore\ k=7{\cdot}5.$$

The equation to BE is

$$y=-3x+7{\cdot}5. \quad . \quad . \quad . \quad . \quad (2)$$

On solving the equations (1) and (2) we find that OD and BE intersect at the point $(\frac{5}{3},\ \frac{5}{2})$.

RÉSUMÉ

1. The angle between the two lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ is given by the formula

$$\tan\theta=\frac{l_1m_2-l_2m_1}{l_1l_2+m_1m_2}.$$

The lines are parallel if $\frac{l_1}{l_2}=\frac{m_1}{m_2}$, and perpendicular if $l_1l_2+m_1m_2=0$.

2. Lines are parallel whose equations differ in the absolute terms only.

3. Two lines are perpendicular if the coefficients of x and y are interchanged in their equations and the sign of one coefficient reversed.

4. The gradient of the line joining the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

5. $y = mx + b$ is the equation to a line of gradient m, making an intercept of b units on the y-axis, or passing through the point (o, b).

6. Two lines are parallel if their gradients are equal, and perpendicular if the product of their gradients is -1. Put symbolically we have

$$\text{(i.)}\quad m = m'.$$
$$\text{(ii.)}\quad mm' = -1$$
$$\text{or}\quad m' = -\frac{1}{m}.$$

EXAMPLES

1. Find the acute angles between the following pairs of lines:

$$(1)\begin{cases} 3x - 2y = 1 \\ 4x + \ \ y = 2. \end{cases}$$
$$(2)\begin{cases} 2x - 3y + 12 = 0 \\ \ \ x - 5y + \ \ 5 = 0. \end{cases}$$
$$(3)\begin{cases} 4x + 6y - 5 = 0 \\ 3x + 5y + 1 = 0. \end{cases}$$

2. Find the angle between the following pairs of lines:

$$(1)\begin{cases} \ \ 4x + 2y = 5 \\ 12x + 6y = 7. \end{cases}$$
$$(2)\begin{cases} \ \ 2x - \ \ 3y = 1 \\ 15x + 10y = 4. \end{cases}$$

3. Make the construction of Article I. for the following pairs of lines:

$$(1)\begin{cases} 4x + 5y = 3 \\ 3x + 2y = 4. \end{cases}$$
$$(2)\begin{cases} 7x - 4y = 1 \\ 5x - 6y = 3. \end{cases}$$

In each case find the size of the angles OM_1L_1 and OM_2L_2. Hence find the angle between the lines and verify by the usual method.

4. Perpendiculars are drawn to the axes from the point $C \equiv (a, b)$.

Find the equations to the diagonals of the rectangle so formed. Hence obtain an expression for the tangent of the angle which they include.

5. Prove that the following pairs of lines are parallel:

$$(1)\begin{cases} 3x - 2y = 4 \\ 6x - 4y = 3. \end{cases}$$

$$(2)\begin{cases} 4x + 7y = 1 \\ 20x + 35y = 6. \end{cases}$$

6. A straight line is drawn through the point (3, 4) parallel to the line $x+2y=5$. Find its equation.

7. Find the equation to the straight line passing through the point (2, −3) and parallel to $4x-5y=2$.

8. A straight line cuts the axes at $A\equiv(a, o)$ and $B\equiv(o, b)$. Find the equation to the straight line which joins the mid points of OA and OB, and hence show that it is parallel to AB.

9. A straight line is drawn through the intersection of the lines $2x+y+1=0$ and $6x-5y=7$, parallel to the line $2x-3y=0$. Find its equation.

10. A straight line is drawn through the origin parallel to the line $\frac{x}{a}+\frac{y}{b}=1$. Find its equation.

11. Prove that the following pairs of lines are perpendicular:

$$(1)\begin{cases} 3x - y = 4 \\ 5x + 15y = 2. \end{cases}$$

$$(2)\begin{cases} 6x + 4y = 7 \\ 28x - 42y = 9. \end{cases}$$

12. A straight line is drawn through the point (1, 1) perpendicular to $5x-7y=2$. Find its equation.

13. A straight line is drawn through the origin perpendicular to $7x+9y=2$. Find its equation.

14. Perpendiculars CA and CB are drawn from the point $C\equiv(a, b)$ to the axes. Find the equation of the perpendicular from the origin to the diagonal AB.

15. In last example if $OACB$ is a square prove by analytical methods that the diagonals are perpendicular.

16. OPQ is a triangle such that $O\equiv(0, 0)$, $P\equiv(3, 3)$ and $Q\equiv(0, 4)$. Find the equations to the altitudes of the triangle through O and Q. Solve them and so find the orthocentre of the triangle.

17. A straight line cuts the axes at $A\equiv(4, 0)$ and $B\equiv(0, 3)$.
P is a variable point on it.
Q is taken on OP so that $OQ=\frac{1}{3}OP$.
Show that the locus of Q is a straight line parallel to AB.

18. A and B are two points on the x and y axes respectively.
OD is perpendicular to AB, and OE bisects angle AOB.
AF is perpendicular to OE.
If $OA=a$ and $OB=b$, obtain the equations to AB, OE, OD, and AF.
Prove that the angle between OE and OD is equal to the angle between AB and AF.

19. A straight line cuts the axes at A and B.
Prove that the median through O of triangle OAB bisects all lines parallel to AB.

(*Hints*: Take the equation to AB in intercept form. The equation to any parallel, PQ, will differ only in the absolute term. Find the equation to OC, where C is the mid-point of AB. Show that the mid-point of PQ lies on AB.)

20. Write down the gradients of the lines joining the following pairs of points:—

(1) (1, 2) and (4, 3).
(2) (−2, 5) and (3, 2·6).
(3) (0, 3) and (−4·2, 0).
(4) (0, 0) and (2, −3·8).
(5) (0, 0) and (5, 4).
(6) (0, 0) and (−5, −4).
(7) (a, o) and (o, b).

21. Find the slope of the lines joining the given pairs of points.

(1) (3·2, 4·5) and (6·5, 7·2).
(2) (0, −1·4) and (−2·2, 1·9).
(3) (0, 0) and (3, −5·6).

22. What is the gradient of all lines perpendicular to (1) $5x-2y=3$ and (2) $4x+7y+10=0$?

23. Find the gradients of the lines joining the pairs of points (2, 1), (4, 4), and (5, −4), (−1, 0). Use the test $m_1m_2=-1$ to prove the lines perpendicular.

24. $ABCD$ is a square, F is the mid-point of AD, and E is taken on AC so that $AE : EC=1:2$. Prove that DE is perpendicular to FC. (Use $m_1m_2=-1$.)

25. $A\equiv(a, o)$, $B\equiv(-a, o)$, and $P\equiv(x', y')$.
If PA is perpendicular to PB prove that $x'^2+y'^2=a^2$. (Use $m_1m_2=-1$.)

26. Use Article I. to find $\tan\theta$ in the case of the lines $y=m_1x+b_1$ and $y=m_2x+b_2$.

27. A straight line cuts the axes at A and B, and C is the mid-point of AB. Perpendiculars CM and CN are drawn to the axes. If P is a variable point such that the area of the quadrilateral $ONPM$ is equal to $\Delta\, OAB$, show that the locus of P is a straight line parallel to AB.

28. KL is taken on XX' of length l units, and MN is taken

on YY' of length m units. P is a variable point such that $\Delta\, KPL - \Delta\, MPN$ is constant. Prove that the locus of P is a straight line perpendicular to $lx+my+n=0$.

29. PN is perpendicular to XX' and PM to YY'.

(i.) Find the locus of P if the perimeter of the rectangle $ONPM$ is constant.

(ii.) Find the locus of P if the length of the rectangle always exceeds its width by a given amount. Prove that the two loci are perpendicular lines.

30. $OADB$ is a trapezium having $A\hat{O}B = OBD = 90°$. If $OA = 8$ units, $OB = 7$ units, and $DB = 3$ units, find the angle between OD and AB.

31. Find the angle between the lines $x+y-5=0$ and $2x-2y+3=0$. Find also the area of the quadrilateral which the lines form with the axes.

32. ABC is a triangle. BE is perpendicular to AC, and CF to AB. If CF cuts BE at H, prove analytically that $EC \cdot EA = EH \cdot EB$. (See Article I. Example 4.)

33. CAB is a right-angled triangle. A line perpendicular to the hypotenuse AB meets CB at Q and CA at P. Prove BP perpendicular to AQ.

34. Find the orthocentre of the triangle whose sides are the lines $x=y$, $x=2y$ and $x+2y=8$.

35. Find the co-ordinates of the foot of the perpendicular from the origin to the line $\dfrac{x}{a}+\dfrac{y}{b}=1$.

If the line cuts the axes at A and B, and if D is the foot of the perpendicular from O to AB, find the lengths of OD, AD, and BD. Hence prove that $OD^2 = AD \cdot DB$.

36. Write down the equation to lines having gradients and y-intercepts as follows:

(i.) gradient $\frac{2}{3}$; intercept 4.
(ii.) gradient $-\frac{5}{6}$; intercept 3.
(iii.) gradient $-\cdot 8$; intercept -2.
(iv.) gradient $1\cdot 7$; intercept zero.

37. Find the mid-point of the length intercepted by the axes on the line $y=mx+b$.

38. Find the locus of the mid-points of the lengths intercepted by the axes on a series of parallel lines. (*N.B.*: All the lines have the same gradient.)

39. What are the co-ordinates of D, the foot of the perpendicular from O to the line $2x+y=10$?

Find another point P on the given line and draw from it perpendiculars PN and PM to the axes. Prove that MN subtends a right angle at D. (Hint: Use $m\,m_1 = -1$.)

40. Points B and B' are taken on the y-axis so that $OB = OB'$. The gradient of a line through B is m. If $OB = b$ write down the equation to a line through B' perpendicular to that through B. Where do the two lines intersect ?

41. A straight line cuts the axes at $A \equiv (a, o)$ and $B \equiv (o, b)$. The bisector of $X\hat{O}Y$ is drawn.

(i.) Work out the equations to the perpendiculars from A and B to the bisector.

(ii.) Find the co-ordinates of D and E, the feet of the perpendiculars.

(iii.) Parallels are drawn through D and E to the axes. Prove that two of them intersect on AB.

42. A straight line cuts the axes at A and B.

C is the mid-point of AB.

A perpendicular BD is drawn from B to OC.

(i.) If $A \equiv (a, o)$ and $B \equiv (o, b)$, what is the gradient of BD ?

(ii.) Hence obtain the equation to BD and find the point E where it cuts XX'.

Prove $OA \cdot OE = OB^2$.

43. If a triangle ABC is right-angled at C find an expression for the tangent of the angle between the lines joining C to the corners of the square on the hypotenuse as in the figure of Pythagoras' theorem, the sides of the triangle being a units and b units in length.

44. *ABC is a triangle right-angled at C.*

B is joined to any point P on AC or AC produced.

AQ is perpendicular to PB and cuts PQ at R.

Prove that the system of lines PR are all parallel to one another.

(Hint : Use intercept form of equations to straight lines BP and AR. Prove the gradient of PR constant.)

45. A point B is taken on the y-axis. Through it are drawn a pair of perpendicular lines. Parallels are drawn to these lines through the origin so forming a rectangle. Prove analytically that the variable diagonal always bisects OB.

(Hint : Use the gradient form of the equation to a straight line.)

46. Two straight lines are drawn, one through the point (0, 2) and the other through the point (2, 1). If the intercepts made by the axes on the first are three times those made by the second find the equations to the lines.

47. Three points A, B, and C are taken such that $A \equiv (12, 0)$, $B \equiv (0, 15)$, and $C \equiv (8, 0)$. Parallels are drawn through C to OY and AB. Find the point of intersection of the diagonals of the parallelogram so formed.

48. From any point on a given straight line perpendiculars PM and PN are drawn to the axes. Prove that the locus of the mid-point of MN is a straight line parallel to the given one.

49. $ABCD$ is a square. A variable line is drawn through A. Perpendiculars BP and DQ are drawn to the line Parallels are drawn through P and Q to the sides of the square, so forming a rectangle $PSQR$. Prove that S and R always lie on one or other of the diagonals of the square

50. ABC is an isosceles triangle such that $A \equiv (a, o)$, $B \equiv (o, b)$, and $C \equiv (-a, o)$. Find an expression for the tangent of the angle between the medians through A and C

51. In last example prove that the angle between the altitudes through A and C is equal to the angle between AB and BC

52 $OACB$ is a rectangle such that $A \equiv (a, o)$ and $B \equiv (o, b)$
P and Q are taken on XX' so that OP OQ OB

Find the ratio $\dfrac{\tan BPC}{\tan BQC}$.

53. Through the point C $(0, 3)$ two straight lines are drawn whose gradients are $\frac{3}{5}$ and $-\frac{4}{5}$ respectively. Perpendiculars OA and OB are drawn to these lines What angle does AB subtend at O? Prove it supplementary to $A\hat{C}B$.

54. A straight line of gradient m cuts the axes at A and B.
C is the mid point of AB and OD is perpendicular to AB.
If OB b find the tangent of the angle between OD and OC
If $D\hat{O}C - 45°$ find the slope of AB

55 B and B' are two points on the y axis such that $OB = OB'$
(i) Write down the equation to a line through B of gradient m.
(ii) Find the point P where this line cuts the parallel to the x axis through B'.
(iii) A parallel to the y axis cuts these lines at Q and R What are the co-ordinates of S, the mid point of QR?
(iv) Prove that OS passes through P

56 AB is the hypotenuse of a right-angled triangle ABC. Any straight line parallel to AB cuts CA at P and CB at Q Find the locus of the intersection of AQ and BP

(Hints Take the equation to AB in intercept form and use the fact that the equation to PQ differs from it only in the absolute term)

57. From the vertex B of an isosceles triangle ABC a perpendicular BD is drawn to the base DE is drawn perpendicular to AB. B is joined to F, the mid point of DE Prove BF perpendicular to CE.

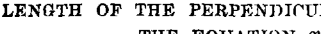
CHAPTER IV

LENGTH OF THE PERPENDICULAR FROM A POINT TO A LINE : THE EQUATION $x \cos \alpha + y \sin \alpha = p$

I. *The expression $lx+my+n$ has the same sign for all points (x, y) on one side of the line $lx+my+n=0$, and has the contrary sign for those on the other side of the line.*

Let AB be the straight line $lx+my+n=0$.

Let $P \equiv (x_1, y_1)$ be any point in the plane of the axes.



Draw PN perpendicular to XX' cutting AB at Q.

Then $Q \equiv (x_1, NQ)$. Now Q lies on AB,

$$\therefore lx_1 + m \cdot NQ + n = 0,$$

$$\therefore lx_1 + n = -m \cdot NQ,$$

$$\therefore lx_1 + my_1 + n = my_1 - mNQ,$$

$$= m(NP - NQ).$$

Hence since m is a constant the sign of lx_1+my_1+n is dependent on that of the difference $NP-NQ$, which is positive or negative according as NP is greater or less than NQ. Now the figure shows us that NP will always be greater than NQ so long as P lies on one side of the line AB, but will be less than NQ when P is on the other side, as at P'. Consequently the sign of lx_1+my_1+n is plus for all points on one side of the line $lx+my+n=0$, and is minus for points on the other side.

When P lies on AB then $NP=NQ$, so that $lx_1+my_1+n=0$ as we know it should be. The line is a sort of neutral zone with regard to the sign of the expression $lx+my+n$.

We have, then, the following procedure. To see whether

two points lie on the same or opposite sides of the line $lx+my+n=0$ substitute their co-ordinates in the expression $lx+my+n$. If like signs result the points are on the same side of the line, but if contrary signs are obtained the points are on opposite sides of it.

EXAMPLE 1.—*Is the point* (2, 5) *on the same side of the line* $3x+4y-12=0$ *as the origin?*

When $x=0$ and $y=0$ then $3x+4y-12=-12$.
When $x=2$ and $y=5$ then $3x+4y-12=+14$.

The point (2, 5) is on the side remote from O.

EXAMPLE 2.—*Classify the following points with regard to the line* $2x+5y=10$:—(1, 2), (−6, 1), (−3, 2), (−3, 4), (2, −1), (−1, −2), (3, 5), *and* (6, −1).

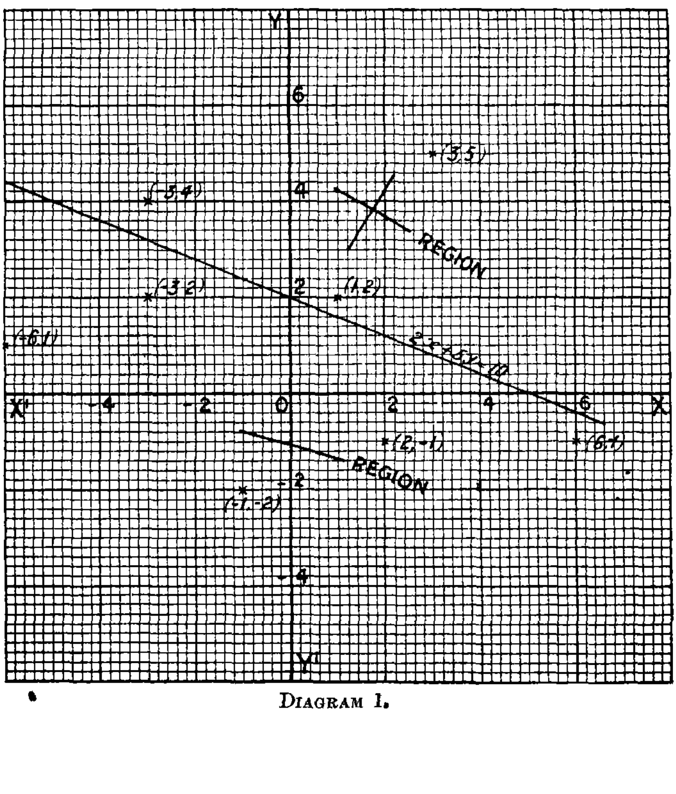


DIAGRAM 1.

We must first write the equation to the straight line thus:

$$2x+5y-10=0.$$

When $x=0$ and $y=0$ then $2x+5y-10=-10$.
When $x=1$ and $y=2$ then $2x+5y-10=+2$.
When $x=-6$ and $y=1$ then $2x+5y-10=-17$.
When $x=-3$ and $y=2$ then $2x+5y-10=-6$.
When $x=-3$ and $y=4$ then $2x+5y-10=+4$.
When $x=2$ and $y=-1$ then $2x+5y-10=-11$.
When $x=-1$ and $y=-2$ then $2x+5y-10=-22$.
When $x=3$ and $y=5$ then $2x+5y-10=+21$.
When $x=6$ and $y=-1$ then $2x+5y-10=-3$.

Hence the points $(-6, 1)$, $(-3, 2)$, $(2, -1)$, $(-1, -2)$, and $(6, -1)$ lie on the origin side of the line, while the points $(1, 2)$, $(-3, 4)$, and $(3, 5)$ lie on the other side.

It has to be remarked that we could have written the equation to the line as follows:

$$-2x-5y+10=0.$$

This is merely the form we chose multiplied throughout by -1. In this event the positive and negative regions will now be interchanged. The signs of all our results will, however, be reversed, so that the classification of our points will in no way be affected. Let us take, for example, the first two points, namely $(1, 2)$ and $(-6, 1)$.

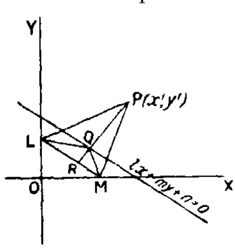
When $x=0$ and $y=0$ then $-2x-5y+10=+10$.
When $x=1$ and $y=2$ then $-2x-5y+10=-2$.
When $x=-6$ and $y=1$ then $-2x-5y+10=+17$.

Hence we have shown as before that $(-6, 1)$ is on the same side as the origin and $(1, 2)$ on the remote side.

At the outset of our work we must therefore prepare our equation and abide by the arrangement throughout.

II. *Find the length of the perpendicular from the point $P\equiv(x', y')$ to the straight line $lx+my+n=0$.*

Mark the points $M\equiv(m, o)$ and $L\equiv(o, l)$.



Then LM is parallel to $lx+my+n=0$ (Chap. II.).

Draw PQR perpendicular to the two parallel lines.

Join PL, PM, QL, and QM.

The quadrilateral $LPMQ$ is the difference of the quadrilaterals $OLPM$ and $OLQM$.

Now $lx'+my'=2$ quad. $OMPL$ (Chap. II Art. III.), and

$-n = 2$ quad. $OMQL$, since Q is a point on the line $lx + my + n = 0$ (Chap. II. Art. III.).

Subtract $\quad \therefore lx' + my' + n = 2$ quad. $LQMP$ (1).

But $\quad 2$ quad. $LQMP = 2 \triangle LPQ + 2 \triangle MPQ.$

$$= PQ \cdot LR + PQ \cdot MR.$$

$$= PQ \cdot LM.$$

$$= PQ\sqrt{l^2 + m^2} \quad . \quad . \quad . \quad (2).$$

Hence by (1) and (2) $PQ\sqrt{l^2 + m^2} = lx' + my' + n$,

$$\therefore p = PQ = \frac{lx' + my' + n}{\sqrt{l^2 + m^2}}.$$

Remarks.—The numerator of this formula is the expression $lx' + my' + n$. Hence by last article p will not only have magnitude, but also a sign depending on which side of the line $lx + my + n = 0$ the point (x', y') lies. The signs of perpendiculars from points on one side of the line will be plus, those from points on the other side minus.

The procedure for finding the length of the perpendicular in any given case is the following. Bring all terms of the equation to the straight line to one side, say the left-hand one. Then substitute (x', y') in the left-hand member of the prepared equation, and put the result over the square root of the sum of the squares of the coefficients of x and y. The actual length of the perpendicular will, of course, in no way depend on the manner in which we write the left-hand member of the equation, but the sign will be affected, contrary signs being obtained for the two arrangements $lx + my + n = 0$ and $-lx - my - n = 0$. In any problem where sign is of importance, that is to say in a question involving the situation of the point relative to the line, we must at the outset decide between the two forms of the equation and adhere to our choice throughout the work.

EXAMPLE 1.—*Find the lengths of the perpendiculars from the origin and from the point* (2, 3) *to the straight line* $3x + 5y - 6$.

We first write the equation thus:

$$3x + 5y - 6 = 0.$$

Let the two perpendiculars be p_0 and p.

(i.) Putting $x=0$ and $y=0$ in the expression $3x+5y-6$ we have

$$p_0=\frac{-6}{\sqrt{3^2+5^2}},$$
$$=-1\cdot03 \text{ (approx.)}.$$

(ii.) Putting $x=2$ and $y=3$ we have

$$p=\frac{15}{\sqrt{3^2+5^2}},$$
$$=2\cdot57 \text{ (approx.)}.$$

The lengths of the perpendiculars are approximately 1·03 units and 2·57 units respectively, and the signs of the results show us that the origin and the point (2, 3) are on opposite sides of the line.

EXAMPLE 2.—*Work out the results of last example, writing the equation to the straight line in the form* $-3x-5y+6=0$.

In this case

$$p_0=\frac{6}{\sqrt{(-3)^2+(-5)^2}}=1\cdot03 \text{ approx.}$$
$$\text{and } p=\frac{-15}{\sqrt{(-3)^2+(-5)^2}}=-2\cdot57.$$

The numerical values are the same. The signs are reversed, but they still show us that the points are on opposite sides of the line.

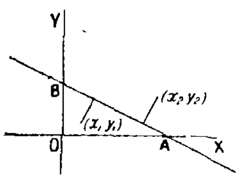
EXAMPLE 3.—*Find the length of the perpendicular from the origin to the line* $lx+my+n=0$.

Let the length of the perpendicular be p_0 units.

Then on putting $x=y=0$ in the expression $lx+my+n$ we obtain

$$p_0=\frac{n}{\sqrt{l^2+m^2}}.$$

EXAMPLE 4.—*Find the locus of a point which moves at a distance of two units from the line* $3x+4y=12$ *and on the origin side of it.*



Let (x_1, y_1) be any position of the variable point.

Write the equation to the line in the form

$$3x+4y-12=0.$$

The length of the perpendicular from the origin is

$$\frac{-12}{\sqrt{9+16}}=-\frac{12}{5}.$$

Perpendiculars from the origin side of the line will therefore be negative.

Hence since we are given that the length of the perpendicular from (x_1, y_1) is 2 units,

$$\frac{3x_1+4y_1-12}{5}=-2,$$
$$\therefore 3x_1+4y_1=2.$$

The locus of (x_1, y_1) is the graph of the equation $3x+4y=2$. It is a straight line parallel to the given one AB.

EXAMPLE 5.—*Obtain the locus of last example when the point moves on the side remote from O.*

Let (x_2, y_2) be a position of the variable point. Using the arrangements of last example, we see that the sign of the perpendicular from (x_2, y_2) must be plus, since that from O is minus,

$$\therefore \frac{3x_2+4y_2-12}{5}=2,$$

$$\therefore 3x_2+4y_2=22.$$

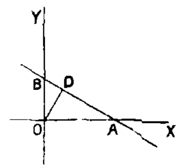
The locus in this case is the line $3x+4y=22$, also a parallel to AB.

III. *An ab initio method of finding the length of the perpendicular from O to a given line $lx+my+n=0$.*

Let AB be the line $lx+my+n=0$.

Draw OD perpendicular to AB.

Then either of the expressions



$$\frac{OD\cdot AB}{2} \text{ or } \frac{OA\cdot OB}{2}$$

measures the area of $\Delta\, OAB$.

$$\therefore OD\cdot AB=OA\cdot OB \quad . \quad . \quad . \quad (1).$$

From the equation $lx+my+n=0$ we have $OA=-\frac{n}{l}$ and $OB=-\frac{n}{m}$,

$$\therefore AB^2=\frac{n^2}{l^2}+\frac{n^2}{m^2}=\frac{n^2(l^2+m^2)}{l^2m^2},$$

$$\therefore AB=\frac{n\sqrt{l^2+m^2}}{lm},$$

$$\therefore \frac{n\sqrt{l^2+m^2}}{lm}\times OD=\frac{n^2}{lm} \text{ (by 1)},$$

$$\therefore OD=\frac{n}{\sqrt{l^2+m^2}}.$$

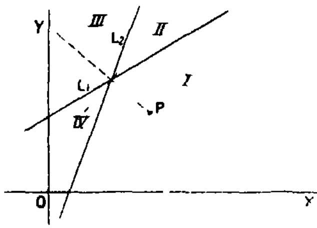
Compare this result with Example 3 of last article.

IV. *Equations to the bisectors of the angles between two lines.*

Let L_1 and L_2 be the two lines, and let their equations be $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ respectively.

The lines divide the plane of the axes into four regions which we have numbered I., II., III., and IV.

It is a well-known geometrical property of the bisector of an angle between two lines that any point on it is equidistant from the lines, so we shall use this fact to obtain the equations of the bisectors.



Consider a point P on the bisector which passes through regions I and III. Suppose that in region I it is, as regards perpendiculars, on the positive side of both lines. Then since in passing into region III the point crosses both lines it will there be on the negative side of each line. Hence in region I. we have $p_1 = p_2$, and in region III $-p_1 = -p_2$, which is the same thing as $p_1 = p_2$.

If therefore $P \equiv (x', y')$, we have on equating perpendiculars for this bisector

$$\frac{l_1x' + m_1y' + n_1}{\sqrt{l_1^2 + m_1^2}} = \frac{l_2x' + m_2'y + n_2'}{\sqrt{l_2^2 + m_2^2}} \qquad (1)$$

Take next the case of a point lying on the other bisector, and suppose it to be in region II. In passing from region I to region II a point crosses the line L_1 but not the line L_2. It therefore follows that the signs of the perpendiculars to L_1 will now be negative, while those to L_2 will remain positive. If we were to take a point in region IV. it is the line L_2 that is crossed in passing from I., so that the signs of the perpendiculars to L_2 are changed, but not these to L_1. In either case, however, the signs of the perpendiculars to L_1 and L_2 are unlike. Hence if $P \equiv (x', y')$ is a point on the bisector which lies in regions II. and IV., then

$$\frac{l_1x' + m_1y' + n_1}{\sqrt{l_1^2 + m_1^2}} = -\frac{l_2x' + m_2y' + n_2}{\sqrt{l_2^2 + m_2^2}} \quad . \quad (2)$$

From results (1) and (2) we see that the equations to the bisectors are

$$\frac{l_1x+m_1y+n_1}{\sqrt{l_1^2+m_1^2}}=\frac{l_2x+m_2y+n_2}{\sqrt{l_2^2+m_2^2}},$$

$$\text{and}\quad \frac{l_1x+m_1y+n_1}{\sqrt{l_1^2+m_1^2}}=-\frac{l_2x+m_2y+n_2}{\sqrt{l_2^2+m_2^2}},$$

or written together

$$\frac{l_1x+m_1y+n_1}{\sqrt{l_1^2+m_1^2}}=\pm\frac{l_2x+m_2y+n_2}{\sqrt{l_2^2+m_2^2}}$$

The equations to the bisectors of the angles between two lines are thus obtained by equating the expressions for the lengths of perpendiculars to the lines from points on the bisectors, in one case the signs being like and in the other unlike.

EXAMPLE.—*Obtain the equations to the bisectors of the angles between the lines $3x-4y=7$ and $12x+5y=15$ and prove that they are at right angles.*

First of all we must write the equations in the necessary shape.

$$3x-4y-7=0 \quad . \quad . \quad . \quad . \quad (1),$$
$$\text{and } 12x+5y-15=0 \quad . \quad . \quad . \quad . \quad (2).$$

The equations to the bisectors are therefore

$$\frac{3x-4y-7}{5}=\frac{12x+5y-15}{13} \quad . \quad . \quad . \quad (3),$$

$$\text{and}\quad \frac{3x-4y-7}{5}=-\frac{12x+5y-15}{13} \quad . \quad . \quad (4).$$

Simplification of results (3) and (4) gives

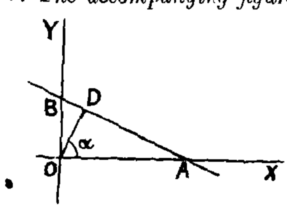
$$21x+77y+16=0,$$
$$\text{and } 99x-27y-166=0.$$

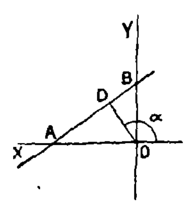
Now the product of the gradients

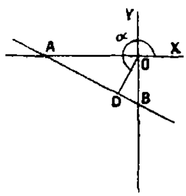
$$=-\tfrac{21}{77}\times\tfrac{99}{27},$$
$$=-1,$$

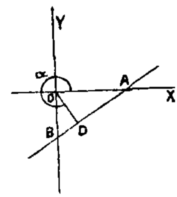
proving that the bisectors are at right angles.

V. *The accompanying figures will show that if we know the*

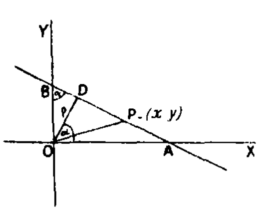








length of OD, the perpendicular from the origin to the line AB, and if we also know the angle α between OX and OD then we can draw AB.



VI. *Find the equation to a straight line at a distance of p units from the origin, and such that the angle between OX and OD, the perpendicular from O, is α.*

Let the straight line cut the axes at A and B, and let $P=(x, y)$ be any point on it. Join OP.

Then $X\hat{O}D=\alpha$, and $O\hat{B}A=90^\circ-D\hat{O}B=\alpha$

Now $$\Delta\, OPB+\Delta\, OPA=\Delta\, OAB,$$

$$\therefore x\cdot OB+y\cdot OA=p\cdot AB,$$

$$\therefore x\cdot\frac{OB}{AB}+y\cdot\frac{OA}{AB}=p,$$

$$\therefore x\cos\alpha+y\sin\alpha=p.$$

This is known as the "perpendicular form" of the equation to a straight line.

EXAMPLE 1.—*Find analytically the intercepts made by the line $x\cos\alpha+y\sin\alpha=p$ on the axes.*

When $y=0$ then $x=\frac{p}{\cos\alpha}$, and when $x=0$ then $y=\frac{p}{\sin\alpha}$.

$$\therefore OA=\frac{p}{\cos\alpha} \text{ and } OB=\frac{p}{\sin\alpha} \text{ as the figure shows.}$$

EXAMPLE 2.—*Find the equation and gradient of OD.*

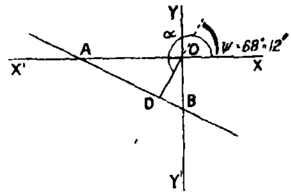
The equation to AB being $x\cos\alpha+y\sin\alpha=p$, that of a line through the origin perpendicular to it is $x\sin\alpha-y\cos\alpha=0$ (Chap. III. Art. III.). The equation to OD is therefore

$$y\cos\alpha=x\sin\alpha \text{ or } y=x\tan\alpha.$$

This last form shows that the gradient is $\tan\alpha$.

EXAMPLE 3.—*Find the size of $X\hat{O}D$ for the line $2x+5y+3=0$.*

(1) The line makes negative intercepts on the axes, and therefore lies as shown in the figure.



The equation to the line as given is

$$2x+5y+3=0,$$

and in perpendicular form is

$$x\cos\alpha+y\sin\alpha=p.$$

Equating the lengths of the intercepts on the axes as derived from these two forms we have

$$\frac{p}{\cos\alpha}=-\frac{3}{2}\text{ (x-intercept).}$$

$$\frac{p}{\sin\alpha}=-\frac{3}{5}\text{ (y intercept).}$$

Divide
$$\therefore \frac{\sin\alpha}{\cos\alpha}=\frac{5}{2},$$

$$\therefore \tan\alpha=2{\cdot}5,$$

$$\therefore \alpha=68°\,12' \text{ or } 248°\,12'.$$

Now the figure shows us that $X\hat{O}D$ is between 180° and 270°,

$$\therefore X\hat{O}D=248°\,12'.$$

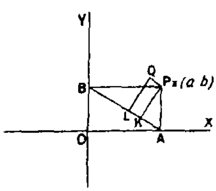
(2) *Alternative Method.*—The equation to OD is $5x-2y=0$. Its gradient is therefore $2{\cdot}5$, so that its slope $\psi=68°\,12'$.

To obtain $X\hat{O}D$ we must add 180° to ψ (see figure).

$$\therefore X\hat{O}D=248°\,12' \text{ as before.}$$

VI. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*From a point $P\equiv(a, b)$ perpendiculars PA and PB are drawn to the axes. Find the distance of P from the diagonal AB of the rectangle $OAPB$.*



The equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1,$$

$$\therefore bx+ay-ab=0.$$

The distance of $P\equiv(a, b)$ from this line is

$$\frac{ab+ab-ab}{\sqrt{a^2+b^2}}=\frac{ab}{\sqrt{a^2+b^2}}.$$

We note that for the distance of O from AB we have

$$p_0=\frac{-ab}{\sqrt{a^2+b^2}}.$$

The difference of sign in these results is due to the fact that O and P are on different sides of AB.

EXAMPLE 2.—*In the figure to last example L is the mid-point of AB, LQ is perpendicular to AB, PQ is perpendicular to LQ, and PK is perpendicular to AB. Find the area of the rectangle $KLQP$.*

The equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

The equation to the line LQ which is perpendicular to it is of the form

$$\frac{x}{b}-\frac{y}{a}=k \text{ (Chap. III. Art. III.).}$$

But $$L\equiv\left(\frac{a}{2},\frac{b}{2}\right) \text{ lies on } LQ,$$

$$\therefore \frac{a}{2b}-\frac{b}{2a}=k,$$

$$\therefore \frac{x}{b}-\frac{y}{a}=\frac{a}{2b}-\frac{b}{2a},$$

$$\therefore 2ax-2by-a^2+b^2=0.$$

The length of the perpendicular from $P\equiv(a, b)$ to this line

$$=\frac{2a^2-2b^2-a^2+b^2}{\sqrt{4a^2+4b^2}}=\frac{a^2-b^2}{2\sqrt{a^2+b^2}}.$$

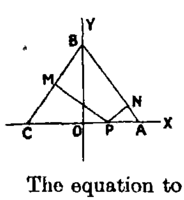
Hence we have $PK=\dfrac{ab}{\sqrt{a^2+b^2}}$ (Ex. 1), and $PQ=\dfrac{a^2-b^2}{2\sqrt{a^2+b^2}}$,

$$\therefore \text{Area of rectangle } KLQP=PK\cdot PQ,$$

$$=\frac{ab(a^2-b^2)}{2(a^2+b^2)}.$$

EXAMPLE 3.—*From any point P on the base AC of an isosceles triangle ABC perpendiculars PM and PN are drawn to the sides AB and BC. Prove*

$$PM+PN=\textit{constant}.$$



Take AC as axis of x and its right bisector as axis of y.

Let $AC=2a$ and $OB=b$.

$$\therefore A\equiv(a, o), B\equiv(o, b), \text{ and } C\equiv(-a, o).$$

The equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1 \text{ or } -bx-ay+ab=0 \quad . \quad . \quad (1).$$

The equation to BC is

$$-\frac{x}{a}+\frac{y}{b}=1 \text{ or } bx-ay+ab=0 \quad . \quad . \quad (2).$$

We write the equations to the sides of the triangle in these forms because we note that P is on the same side of both lines as O, so that the signs of the perpendiculars from O to BA and BC being now plus, those of the perpendiculars from P will also be positive.

Let $P \equiv (x', o)$.

$$\therefore PM = \frac{-bx'+ab}{\sqrt{a^2+b^2}} \text{ and } PN = \frac{bx'+ab}{\sqrt{a^2+b^2}},$$

$$\therefore PN + PM = \frac{2ab}{\sqrt{a^2+b^2}}$$

$$= \text{constant.}$$

RÉSUMÉ

1. The expression $lx+my+n$ is positive or negative according to which side of the line $lx+my+n=0$ the point (x, y) lies.

2. The length of the perpendicular from the point (x', y') to the line $lx+my+n=0$ is given by

$$p = \frac{lx'+my'+n}{\sqrt{l^2+m^2}}.$$

The sign of p will therefore be that of the expression $lx'+my'+n$.

3. The equation to a line whose distance OD from the origin is p units, and is such that $X\hat{O}D = \alpha$ is

$$x \cos \alpha + y \sin \alpha = p.$$

This form is known as " Perpendicular Form."

EXAMPLES

1. Are the points (2, 5) and $(-3, 4)$ on the same or opposite sides of the line $4x-6y=7$?

2. Classify the following points relative to the line $3x=2y-5$.

$(1, 0), (2, 7), (-4, -1), (3, -5), (0, 8)$.

3. Find the length of the perpendicular,

(1) from (2, 1) to the line $4x+3y+5=0$;
(2) from $(-3, 4)$ to the line $2x-5y=7$;
(3) from $(0, -2)$ to the line $y=\cdot 5x+3$;
(4) from $(5, -6)$ to the line $\frac{x}{4}-\frac{y}{2}=1$;
(5) from (5, 12) to the line $3x=7y$.

4. From the point $C \equiv (4, 4)$ perpendiculars CA and CB are drawn to the axes. O is joined to D the mid-point of AC. Find the length of the perpendicular from B to OD.

5. Points B and B' are taken on the y-axis so that $B \equiv (0, 4)$ and $B' \equiv (0, -4)$. Through B' a line of gradient $\frac{1}{2}$ is drawn, and through B a perpendicular BD is drawn to this line. Find the area of $\Delta\, BDB'$, by obtaining the lengths of BD and $B'D$ from the perpendicular formula. Verify the result by finding the distance of D from YY'.

6. $OACB$ is a square such that $C \equiv (c, c)$.
A parallel is drawn through O to the diagonal AB.
Perpendiculars are drawn from A and B to this parallel, so forming a rectangle.
Prove analytically that the rectangle is equal in area to the square.

7. $OACB$ is a rectangle such that $C \equiv (5, 3)$.
From B a perpendicular is drawn to OC.
Perpendiculars are drawn from A to this line and to OC.
Find the area of the rectangle so formed.

8. Find the area of the rectangle of last example if $C \equiv (a, b)$.

9. A point P moves at a distance of 3 units from the line $4x + 3y = 5$ and on the side remote from O. Find its locus and draw it.

10. A point moves at a distance of 5 units from the line $x + y = 6$. Prove that it can lie on either of two straight lines.

11. Find the area of the triangle whose vertices are the origin and the points (2, 5) and (6, 3).

12. Find the area of the triangle whose vertices are at the points $A \equiv (5, 0)$, $B \equiv (0, 2)$, and $C \equiv (3, 4)$.
(*Hint.*—Take the equation to AB in intercept form.)

13. Find the area of the triangle whose vertices are $A \equiv (1, 1)$, $B \equiv (6, 2)$, and $C \equiv (3, 5)$.
Verify the result by the method of Chap. I.
(*Hint.*—Find the equation to AB as in Chap. II. Art. VI. Ex. 1.)

14. Write down the equations to the bisectors of the angles between the following pairs of lines:

$$(1)\begin{cases} 2x + 5y + 3 = 0. \\ 3x - \ \ y + 4 = 0. \end{cases}$$

$$(2)\begin{cases} \dfrac{x}{3} + \dfrac{y}{2} = 1. \\ y = \frac{3}{4}x + 4. \end{cases}$$

$$(3)\begin{cases} x = y. \\ 3x + 4y = 0. \end{cases}$$

Draw the system of No. 2 on squared paper.

15 OAB is a triangle such that $A \equiv (4, 0)$ and $B \equiv (0, 3)$. Work out the equations to the bisectors of the angles OAB and OBA. Show that they intersect on the bisector of $X\hat{O}Y$

16 Find the angle between OX and the perpendicular from O to the line $2x+5y-10$.

17. From the point $C \equiv (6, -4)$ perpendiculars CA and CB are drawn to the axes. OD is drawn perpendicular to AB Find the size of the angle α as regards the line AB. (See 3 of Resume.)

18. With centre O and a radius of 2 units describe a circle
Draw a radius OP such that the angle XOP 50°
Draw the tangent at P and write down its equation in perpendicular form.

19. $ABCD$ is a square. E is any point in BC.
BF and DG are perpendiculars to AE Prove $BF \cdot DG$ $BE : AB$.

20 $ABCD$ is a rectangle E is any point on BC Through D a parallel is drawn to AE, and to it AG and EF are perpendiculars. Prove that the area of the rectangle $AEFG$ is constant

21. A straight line cuts the axes at $A \equiv (a, o)$ and $B \equiv (o, b)$
A square $ABCD$ is described on AB so as to be turned away from O Prove analytically that the distance of its centre E from AB is

$$\frac{\sqrt{a^2+b^2}}{2}.$$

(*Note.*—E is the mid point of AC or BD)

22 $ABCD$ is a square whose sides are of unit length
A variable line through C cuts AB at P and AD at Q.
Prove that the length of the perpendicular from A to PQ is

$$\frac{AP+AQ}{PQ}.$$

23. Prove the theorem of Art. I as follows:—

(a) Draw the quadrilateral $OLPM$ as in Art. II. placing P
  (1) on the side of $lx+my+n=0$ remote from O, and
  (2) on the same side of the line as O.

(b) Take Q at a convenient point on the line, and consider the difference of the quadrilaterals $OLPM$ and $OLQM$.

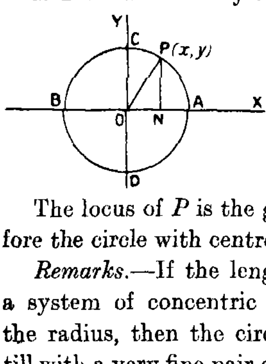
24. Find the length of OD, the perpendicular from O to the line $lx+my+n=0$ by constructing the quadrilateral $OLDM$ and obtaining its area in two ways.

CHAPTER V

EQUATION TO A CIRCLE WHOSE CENTRE IS THE ORIGIN: INTERSECTION OF A STRAIGHT LINE AND A CIRCLE

Definition.—A circle is a plane figure enclosed by a line called its circumference, all points on the latter being equidistant from a certain internal point called the centre.

I. *Find the locus of a point which moves at a distance of a units from the origin.*



Let $P \equiv (x, y)$ be a point such that $OP = a$.

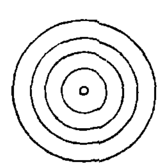
Draw PN perpendicular to XX'.

Then $ON^2 + NP^2 = OP^2$,

$$\therefore x^2 + y^2 = a^2.$$

The locus of P is the graph of this equation, which is therefore the circle with centre 0 and radius of a units.

Remarks.—If the length of a be varied, the result will be a system of concentric circles. If we continuously decrease the radius, then the circles will become smaller and smaller, till with a very fine pair of compasses we should have a mere speck. Hence ultimately, when $a = 0$ we arrive at a point, namely the origin. A degenerate circle of this kind is called a point circle. We say, therefore, that $x^2 + y^2 = 0$ is a point circle at the origin. (See adjoined figure.)



Again, we can write the equation $x^2 + y^2 = a^2$ as follows,

$$y^2 = a^2 - x^2$$
$$\therefore y = \pm \sqrt{a^2 - x^2}.$$

This shows that when any value is assigned to x there are two corresponding values of y, which are numerically equal, *but have opposite signs. The two points* obtained therefrom lie on either side of the x-axis and at equal distances from it, one directly above the other. Further, x can never be numerically greater than a, for then $a^2 - x^2$ would be negative, and there is no real square root to a negative number. Similar remarks will apply when x and y are interchanged in the discussion.

EXAMPLE 1.—*Draw the graph of* $x^2 + y^2 = 4$.

Comparing with the equation $x^2 + y^2 = a^2$, we see that the equation is that of a circle whose centre is O.

In this case $a^2 = 4$, so that the radius of the circle is 2 units.

We have $$y = \pm\sqrt{4 - x^2}.$$

Note that when $x > 2$ say 3, then $y = \pm\sqrt{-5}$, which are unreal numbers.

x	-2	1	0	1	2
y_1	0	1·73	2	1·73	0
y_2	0	$-1·73$	-2	$-1·73$	0

In the graph observe the relative positions of the points (1, 1·73) and (1, $-1·73$).

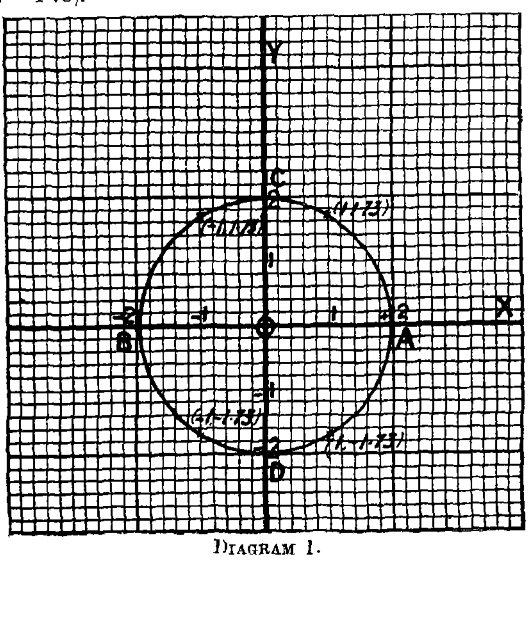


DIAGRAM 1.

Example 2.—*Describe the graph of the equation* $x^2+y^2=5$.

Let $P \equiv (x, y)$ be a point on the locus, then

$$OP^2 = x^2+y^2 = 5 \quad \therefore OP = \sqrt{5}.$$

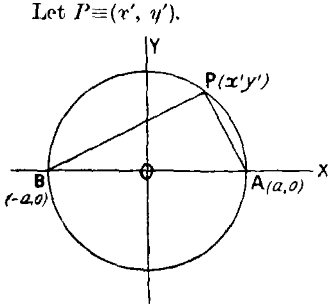
The graph is a circle whose centre is O and radius $\sqrt{5}$, or 2·24 units (approx.).

Example 3.—*Two variable straight lines AP and BP pass through two fixed points A and B, and are at right angles to one another. Find the locus of P.*

Take AB as axis of x and its right bisector as axis of y.

Let $A \equiv (a, o)$ and $B \equiv (-a, o)$.

Let $P \equiv (x', y')$.



The gradient of AP is $\dfrac{-y'}{a-x'}$, The gradient of BP is $\dfrac{-y'}{-a-x'}$ } Chap. III.

But the lines AP and BP are at right angles, therefore the product of their gradients is -1 (Chap. III.).

$$\therefore \frac{y'^2}{-(a+x')(a-x')} = -1,$$

$$\therefore y'^2 = a^2 - x'^2,$$

$$\therefore x'^2 + y'^2 = a^2 \text{ (a constant).}$$

The locus of $P \equiv (x', y')$ is the graph of the equation $x^2+y^2=a^2$, a circle with centre O and radius of a units.

Example 4.—*A point moves so that the sum of the squares of its distances from two fixed points 10 units apart is always 62 sq. units. Find its locus.*

Let the fixed points be A and B (see last figure) such that $AB=10$ units.

Take the same axes as in last example.

Then $\qquad A \equiv (5, 0)$ and $B \equiv (-5, 0)$.

Let $P \equiv (x', y')$.

Now $\qquad AP^2 = (x'-5)^2+y'^2 \quad$ (Chap. I.),

and $\qquad BP^2 = (x'+5)^2+y'^2$.

But $\qquad AP^2+BP^2 = 62$,

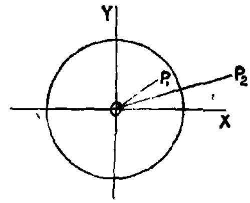
$$\therefore (x'-5)^2+y'^2+(x'+5)^2+y'^2 = 62,$$

$$\therefore x'^2+y'^2 = 6.$$

The locus of $P \equiv (x', y')$ is the circle whose equation is $x^2+y^2=6$.

The centre is O and the radius is $\sqrt{6}$ units, or approx. 2·45 units.

II. *The circle $x^2+y^2=a^2$ divides the plane of the axes into two regions, such that the expression $x^2+y^2-a^2$ has opposite signs for points within and points without the circle.*



Let $P\equiv(x', y')$ be a point in the plane of the circle.

Then $\qquad x'^2+y'^2=OP^2.$

Now if P is within the circle OP is less than the radius, and if P is without the circle OP is greater than the radius.

Hence $x'^2+y'^2-a^2=OP^2-a^2$ is negative if P is inside, and positive if P is outside the circle.

As O is a point within the circle we can compare the result of substituting the co-ordinates of the origin in the expression $x^2+y^2-a^2$, with that obtained on substituting those of any point in question.

EXAMPLE.—*Are the points* (2, 1) *and* (5, -3) *within or without the circle* $x^2+y^2=6$?

(i.) When $x=0$ and $y=0$ then $x^2+y^2-6=-6$.
(ii.) When $x=2$ and $y=1$ then $x^2+y^2-6=-1$.
(iii.) When $x=5$ and $y=3$ then $x^2+y^2-6=28$.

Hence (2, 1) is inside and (5, -3) outside the circle, since the origin is within.

III. *Intersections of a straight line and a circle.*

EXAMPLE 1.—*By drawing the circle* $x^2+y^2=5$ (i.e., $y=\pm\sqrt{5-x^2}$) *and the straight line* $x+y=3$ (i.e., $y=3-x$) *find where they intersect.*

$y=\pm\sqrt{5-x^2}.$

x	-2	$-1{\cdot}5$	-1	0	1	1·5	2
y	±1	$\pm1{\cdot}66$	±2	$\pm2{\cdot}24$	±2	$\pm1{\cdot}66$	±1

$y=3-x.$

x	-3	-2	-1	0	1	2	3
y	6	5	4	3	2	1	0

We see that the straight line and circle intersect at the points (1, 2) and (2, 1). (Diagram 2.)

These points must therefore appear in the graph table of each, as on inspection we find they do.

Hence the co-ordinates of the points of intersection of the straight line and the circle, are the values of x and y which satisfy both their equations.

Verification by solving the equations

$$x^2+y^2=5 \quad . \quad . \quad . \quad . \quad (1),$$
$$\text{and } x+y=3 \quad . \quad . \quad . \quad . \quad (2).$$

From (2) we have $\quad y=3-x.$
Substitute this value of y in (1).

$$\therefore x^2+(3-x)^2=5,$$
$$\therefore x^2-3x+2=0,$$
$$\therefore (x-1)(x-2)=0.$$
$$\therefore x=1 \text{ or } 2.$$
$$\therefore y=2 \text{ or } 1 \text{ (using equation 2).}$$

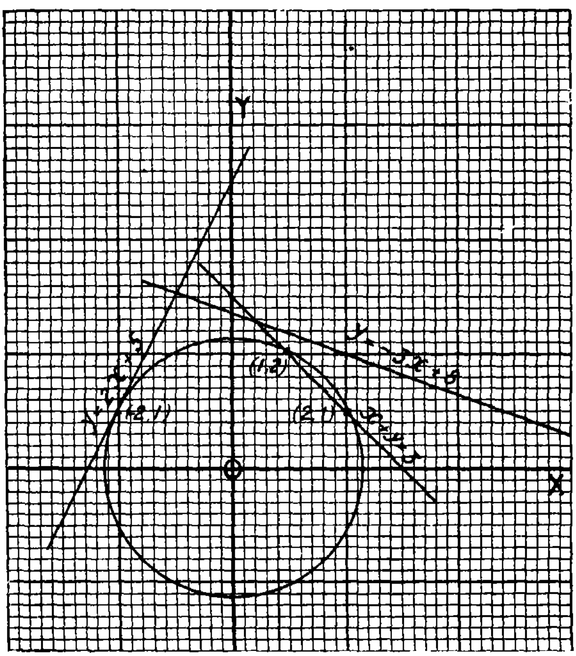


DIAGRAM 2.

EXAMPLE 2.—*By drawing the graphs of their equations, find where the straight line* $y=2x+5$ *cuts the circle* $x^2+y^2=5$.

The circle was drawn for last example. (See Diagram 2.)

$y=2x+5.$

x	-3	-2	-1	0	1	2
y	-1	$+1$	3	5	7	9

This straight line is shown in the same diagram.

We see that the straight line is the tangent to the circle at the point $(-2, 1)$. This point appears in the graph chart of both the straight line and the circle.

Verification by solving their equations:—

As before the co-ordinates of their points of intersection will be the values of x and y which satisfy both their equations.

Solve then the equations

$$x^2+y^2=5 \quad . \quad . \quad . \quad . \quad (1),$$
$$\text{and } y=2x+5 \quad . \quad . \quad . \quad (2).$$

Substitute in (1) the value of y in (2).

$$\therefore x^2+(2x+5)^2=5,$$
$$\therefore x^2+4x+4=0,$$
$$\therefore (x+2)^2=0.$$

Thus both values of x are equal, being each -2.

Equation (2) gives the corresponding values of y as each $+1$.

The straight line therefore cuts the circle in two points coincident with $(-2, 1)$. This is in conformity with the geometrical definition of a tangent.

EXAMPLE 3.—*Find by drawing them, where the straight line $y=-3x+8$ cuts the circle of last two examples.*

$y=-3x+8.$

x	-2	-1	0	1	2
y	14	11	8	5	2

(See Diagram 2.)

The straight line does not cut the circle.

Verification by solving their equations:—

As before we obtain their points of intersection by solving the equations

$$x^2+y^2=5 \quad . \quad . \quad . \quad . \quad (1),$$
$$\text{and } y=-3x+8 \quad . \quad . \quad . \quad . \quad (2).$$

Substitute from (2) in (1)

$$\therefore x^2+(-3x+8)^2=5,$$
$$\therefore 10x^2-48x+59=0,$$
$$\therefore x=\frac{48\pm\sqrt{-56}}{20}.$$

Now $\sqrt{-56}$ is unreal, therefore the roots of the equation $10x^2-48x+59=0$ are imaginary.

The two graphs have therefore no common points in the ordinary geometrical sense, but to preserve continuity of statement we say that the straight line cuts the circle in two imaginary points, which is the geometrical equivalent of the statement that the roots of the equation $10x^2-48x+59=0$ are unreal.

Summarising the results of these three examples we conclude that a straight line cuts a circle in two points real (Example 1), coincident (Example 2), or imaginary (Example 3).

IV. *A straight line cuts a circle in two points, real, coincident, or imaginary.*

Let the equation to the circle be

$$x^2+y^2=a^2 \quad . \quad . \quad . \quad . \quad (1),$$

and that of the straight line be

$$y=mx+b \quad . \quad . \quad . \quad . \quad (2).$$

Their points of intersection will be obtained by solving their equations, just as in the case of two straight lines.

Substitute from (2) in (1) for y,

$$\therefore x^2+(mx+b)^2=a^2,$$

$$\therefore x^2+m^2x^2+2bmx+b^2=a^2,$$

$$\therefore (1+m^2)x^2+2bmx+(b^2-a^2)=0 \quad . \quad . \quad (3).$$

This is a quadratic equation giving the abscissae of the points of intersection.

The corresponding values of the ordinates are found by substituting in equation (2) the values of x found in (3).

There are therefore two points of intersection corresponding to the two values of x found in equation (3).

As the two roots of equation (3) may be real, coincident, or imaginary, it follows that the two points of intersection may be real, coincident (tangent), or imaginary.

EXAMPLE 1.—*Find where the straight line* $3x+4y-12$ *cuts the circle* $x^2+y^2=30$.

We solve the equations

$$x^2+y^2=30 \quad . \quad . \quad . \quad (1),$$

$$\text{and } 3x+4y-12 \quad . \quad . \quad . \quad (2).$$

Write (2) thus $\quad y=\dfrac{12-3x}{4}$.

Then substitute for y in (1)

$$\therefore x^2+\frac{(12-3x)^2}{16}=30.$$

Whence $\quad 25x^2-72x-236=0.$

$$\therefore x=4{\cdot}80 \text{ or } -1{\cdot}92 \text{ (approx.)}$$

By (2) $\quad y=-{\cdot}60 \text{ or } 4{\cdot}44$ (approx.).

The points of intersection are $(4{\cdot}80, -{\cdot}60)$ and $(-1{\cdot}92, 4{\cdot}44)$

EXAMPLE 2.—*Prove that the straight line $2x-5y=29$ touches the circle $x^2+y^2=29$.*

We must show that the two solutions to the equations

$$x^2+y^2=29 \quad . \quad . \quad . \quad . \quad (1),$$
$$\text{and } 2x-5y=29 \quad . \quad . \quad . \quad (2),$$

are coincident.

For variety we shall eliminate x.

Write (2) thus $\qquad x=\dfrac{5y+29}{2}.$

Substitute for x in (1)

$$\therefore \frac{(5y+29)^2}{4}+y^2=29.$$

Whence on simplifying and dividing out 29 we have

$$y^2+10y+25=0.$$
$$\therefore (y+5)^2=0.$$

Thus the ordinates of both points of intersection are -5.

From equation (2) the abscissae of both the points are 2.

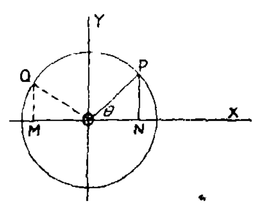
The straight line therefore cuts the circle in two points coincident with $(2, -5)$, or in other words is the tangent there.

V. *If $P\equiv(x, y)$ is a point on the circle $x^2+y^2=a^2$ then $x=a\cos\theta$ and $y=a\sin\theta$ where $\theta=X\hat{O}P$.*

From the figure we have

$ON=OP\cos\theta$ and $NP=OP\sin\theta$,

$\therefore x=a\cos\theta$ and $y=a\sin\theta$.



Note.—$a\cos\theta$ and $a\sin\theta$ are usually called the polar co-ordinates of P. The angle θ is called a vectorial angle.

EXAMPLE 1.—*Find the polar co-ordinates of the point Q $(-1{\cdot}6, 1{\cdot}2)$ on the circle $x^2+y^2=4$* (see last figure).

We have $\qquad \tan\theta=\tan XOQ=-\dfrac{1{\cdot}2}{1{\cdot}6}=-{\cdot}75.$

$$\therefore \theta=143^\circ\ 8'.$$

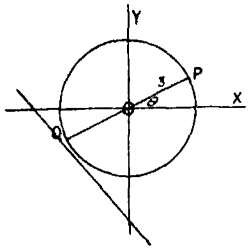
But $r=2$.

$$\therefore Q\equiv(2\cos 143^\circ\ 8', 2\sin 143^\circ\ 8').$$

EXAMPLE 2.—*Any diameter PQ of the circle $x^2+y^2=9$ is drawn. From its ends perpendiculars are drawn to the line $3x+2y+12=0$. Prove that their sum is constant.*

Let $X\hat{O}P=\theta$.

$$\therefore P\equiv(3\cos\theta, 3\sin\theta) \text{ and } Q\equiv(-3\cos\theta, -3\sin\theta).$$



If p_1 and p_2 be the lengths of the perpendiculars, then

$$p_1=\frac{9\cos\theta+6\sin\theta+12}{\sqrt{13}}\quad\text{(Chap. IV.)},$$

and $$p_2=\frac{-9\cos\theta-6\sin\theta+12}{\sqrt{13}}.$$

$\therefore\ p_1+p_2=\frac{24}{\sqrt{13}}=$ constant for all positions of PQ.

VI. **The following properties of quadratic equations will be used hereafter.**

If x_1 and x_2 are the roots of the equation

$$ax^2+bx+c=0,$$

then (1) $$x_1+x_2=-\frac{b}{a}$$

and (2) $$x_1x_2=\frac{c}{a}.$$

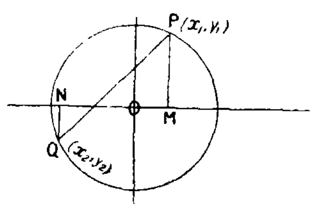
If the roots are equal, then

(3) $$b^2=4ac.$$

EXAMPLE 1.—*The straight line $y=x+1$ cuts the circle $x^2+y^2=6$ in two points P and Q. Perpendiculars PM and QN are drawn from P and Q to XX'. Prove that $OM\cdot ON=2\cdot5$.*

The straight line $y=x+1$ cuts the circle $x^2+y^2=6$ in two points whose abscissae are given by the equation

$$x^2+(x+1)^2=6.$$
$$\therefore\ 2x^2+2x-5=0.$$



Let x_1 and x_2 be the roots of this equation.

Then $x_1x_2=-\frac{5}{2}$.

(Property (2) quoted above.)

Hence $OM\cdot ON=2\cdot5$.

EXAMPLE 2.—*In last example find the mid-point of PQ.*

Let $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$, and the mid-point $R\equiv(x', y')$.

Then $$x'=\frac{x_1+x_2}{2}\text{ and }y'=\frac{y_1+y_2}{2}\quad\text{(Chap. I.)}.$$

Now the abscissae of P and Q are obtained from the equation $2x^2+2x-5=0$ of last example.

By property (1) of quadratic equations we have

$$x_1+x_2=-1.$$

$$\therefore x'=\frac{x_1+x_2}{2}=-\tfrac{1}{2}.$$

We can find y' by eliminating x between the two equations, and going through work similar to that for obtaining x'; or we may use the fact that (x', y') lies on the line PQ. Hence from its equation $y=x+1$ we have when $x=-\frac{1}{2}$, $y=\frac{1}{2}$.

The mid-point therefore is $(-\frac{1}{2}, \frac{1}{2})$.

EXAMPLE 3.—*Find the co-ordinates of the mid-point of the chord cut off from the straight line $y=mx+b$ by the circle $x^2+y^2=a^2$.*

Let $P\equiv(x_1, y_1)$ and $Q\equiv(x_2, y_2)$ be the extremities of the chord.

Let $R\equiv(x', y')$ be the mid-point of PQ.

By eliminating y between the two equations we obtain a quadratic, giving the abscissae of P and Q the points of intersection of the line and the circle.

We have $$x^2+(mx+b)^2=a^2.$$

Whence $(1+m^2)x^2+2bmx+(b^2-a^2)=0.$ (See Art. IV.)

Now x_1 and x_2 are the roots of this equation

$$\therefore x_1+x_2=-\frac{2bm}{1+m^2} \quad \text{(Property (1) of quadratic equations.)}$$

$$\therefore x'=\frac{x_1+x_2}{2}=-\frac{bm}{1+m^2}.$$

But (x', y') lies on the line $y=mx+b$.

$$\therefore y'=mx'+b.$$

$$\therefore y'=-\frac{bm^2}{1+m^2}+b.$$

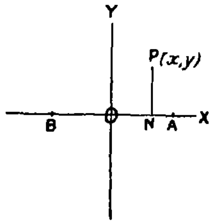
$$=\frac{b}{1+m^2}.$$

$$\therefore R\equiv\left(-\frac{bm}{1+m^2}, \frac{b}{1+m^2}\right).$$

VII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*A and B are two fixed points such that $AB=6$ units. PN is perpendicular to AB. P moves so that $PN^2=AN\cdot NB$. Find the locus of P.*

Take AB as axis of x and its right bisector as axis of y. Then $A\equiv(3, 0)$ and $B\equiv(-3, 0)$.



Let $P \equiv (x, y)$.

Since $PN^2 = AN \cdot NB$.

$$\therefore PN^2 = (OA - ON)(OB + ON)$$
$$= OA^2 - ON^2 \text{ (since } OB = OA).$$
$$\therefore y^2 = 9 - x^2.$$
$$\therefore x^2 + y^2 = 9.$$

The locus of P is a circle with centre O and a radius of 3 units.

EXAMPLE 2.—*A variable line of constant length c units cuts the axes at P and Q. Find the locus of the mid-point of PQ.*

Let $P \equiv (p, o)$, $Q \equiv (o, q)$, and R the mid-point of $PQ \equiv (x, y)$.

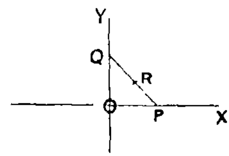
Then $x = \frac{p}{2}$ and $y = \frac{q}{2}$.

That is, $p = 2x$ and $q = 2y$.

But $OP^2 + OQ^2 = PQ^2$,

or $p^2 + q^2 = c^2$.

$$\therefore 4x^2 + 4y^2 = c^2.$$
$$\therefore x^2 + y^2 = \frac{c^2}{4}.$$



The locus of R is a circle whose centre is O and whose radius is $\frac{c}{2}$ units in length.

EXAMPLE 3.—*A circle having O as centre is described so as to touch the straight line $y = \frac{1}{2}x + 3$. Find the length of its radius.*

Let the length of the radius be r units.

(i.) Then the equation to the circle will be $x^2 + y^2 = r^2$.

The straight line $y = \frac{1}{2}x + 3$ cuts the circle at two points whose abscissae are given by the equation

$$x^2 + (\tfrac{1}{2}x + 3)^2 = r^2.$$

whence $5x^2 + 12x + (36 - 4r^2) = 0$.

Now the roots of this equation are equal since the line touches the circle, therefore by Article VI., Property 3 of quadratic equations, we have

$$144 = 20(36 - 4r^2).$$
$$\therefore r^2 = 7 \cdot 2.$$
$$\therefore r = \sqrt{7 \cdot 2}$$
$$= 2 \cdot 68 \text{ nearly.}$$

(ii.) We could also proceed thus. The perpendicular from the centre of a circle to a tangent is the radius to the point of contact.

$$\therefore r = \frac{3}{\sqrt{\frac{1}{4} + 1}} = \frac{6}{\sqrt{5}} = \sqrt{7 \cdot 2} \text{ as before. (See Chap. IV.)}$$

EXAMPLE 4.—*Prove that two tangents of gradient $\frac{2}{3}$ can be drawn to the circle $x^2+y^2=6$.*

Let the equation to any tangent of gradient $\frac{2}{3}$ be

$$y=\tfrac{2}{3}x+b \quad \text{(Chap. III.)}.$$

Then the line $y=\frac{2}{3}x+b$ cuts the circle at points whose abscissae are given by $x^2+(\frac{2}{3}x+b)^2=6$.

$$\therefore x^2+\tfrac{4}{9}x^2+\tfrac{4}{3}bx+b^2-6=0.$$

$$\therefore 13x^2+12bx+9(b^2-6)=0.$$

But the line is a tangent, therefore the roots of this equation are equal.

$$\therefore 144b^2=468(b^2-6) \quad \text{(Art. V.)}$$

$$\therefore b^2=\frac{78}{9}.$$

Hence b may have one of two values $\frac{\sqrt{78}}{3}$ or $-\frac{\sqrt{78}}{3}$.

The lines drawn through the points $\left(o, +\frac{\sqrt{78}}{3}\right)$ and $\left(o, -\frac{\sqrt{78}}{3}\right)$ having a gradient $\frac{2}{3}$ will touch the given circle.

Their equations are $$y=\frac{2}{3}x+\frac{\sqrt{78}}{3},$$

$$\text{and } y=\frac{2}{3}x-\frac{\sqrt{78}}{3}.$$

EXAMPLE 5.—*Prove that two tangents can be drawn to the circle $x^2+y^2=4$ through the point $(5, 3)$, and find their slopes.*

Let the equation to any tangent through the point $(5, 3)$ be

$$y=mx+b.$$

$$\therefore 3=5m+b \quad . \quad . \quad . \quad (1).$$

To find the point of contact we solve the equations

$$x^2+y^2=4$$

$$\text{and } y=mx+b.$$

We have $$x^2+(mx+b)^2=4.$$

$$\therefore (1+m^2)x^2+2bmx+(b^2-4)=0.$$

The roots must be equal

$$\therefore 4b^2m^2=4(1+m^2)(b^2-4).$$

Divide out 4 and simplify.

$$\therefore 4m^2-b^2+4=0.$$

But by (1) $b=3-5m$.

$$\therefore 4m^2-(3-5m)^2+4=0.$$

$$\therefore 21m^2-30m+5=0.$$

$$\therefore m=1{\cdot}2359 \text{ or } {\cdot}1926 \text{ (approx.)}.$$

These are the possible gradients of the tangent.

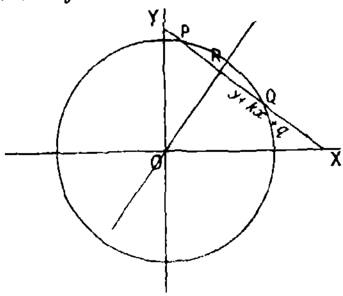
From this we see that we can draw two tangents to the circle through the point (5, 3).

The gradient of one is 1·2359 and of the other ·1926.

Hence we have $\tan\psi = 1{\cdot}2359$ or $\cdot 1926$.

The slopes of the tangents are 51° 1′ and 10° 54′.

EXAMPLE 6.—*Find the locus of the mid-points of a system of parallel chords of a circle.*



Let the equation to the circle be

$$x^2+y^2=a^2.$$

Since the chords of the circle are parallel they will all have the same gradient.

Let it be k.

Let $R\equiv(x', y')$ be the mid-point of any chord of the system PQ whose equation will therefore be

$$y=kx+q.$$

Now we showed in Art. VI., Example 3, that the mid-point of PQ is

$$\left(-\frac{qk}{1+k^2}, \frac{q}{1+k^2}\right).$$

Hence $$x'=-\frac{qk}{1+k^2} \text{ and } y'=\frac{q}{1+k^2}.$$

Between these two expressions we can eliminate q, a variable depending on which chord of the system is drawn.

We have $$\frac{x'}{k}=-\frac{q}{1+k^2},$$

$$\therefore \frac{x'}{k}=-y'.$$

The locus of $R\equiv(x', y')$ is the straight line through the origin which is the graph of the equation $y=-\frac{1}{k}x$.

It is perpendicular to the system of chords, whose gradient is k since $k\times -\frac{1}{k}=-1$ (Chap. III.).

EXAMPLE 7.—*If from the centre of a circle a perpendicular be drau to a chord, it bisects the chord.*

Take the circle and chord PQ of last example.

Then as before

$$R\equiv\left(-\frac{qk}{1+k^2}, \frac{q}{1+k^2}\right).$$

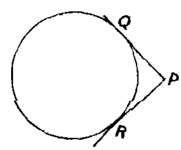
Since the equation to PQ is $y=kx+q$, that of the line through O perpendicular to it is $y=-\frac{1}{k}x$.

Solve these equations to find the point of intersection.

We have
$$-\frac{1}{k}x=kx+q.$$
$$\therefore\ x=-\frac{qk}{1+k^2}.$$
$$\therefore\ y=-\frac{1}{k}x.$$
$$=\frac{q}{1+k^2}.$$

The point of intersection is $\left(-\frac{qk}{1+k^2},\ \frac{q}{1+k^2}\right)$, and is therefore the mid-point R.

EXAMPLE 8.—*Find the locus of the intersection of tangents to a circle which are at right angles to one another.*



Let PQ and PR be perpendicular tangents to the circle $x^2+y^2=a^2$.

Let $P\equiv(x',\ y')$.

Let $y=mx+b$ be the equation to one of the tangents.
$$\therefore\ y'=mx'+b \quad . \quad . \quad (1).$$

Solving the equations $x^2+y^2=a^2$ and $y=mx+b$ in order to find the intersections of the line and circle we have, as in Article IV.,
$$(1+m^2)x^2+2bmx+(b^2-a^2)=0.$$

The roots of this equation are equal since the line is a tangent,
$$\therefore\ 4b^2m^2=4(1+m^2)(b^2-a^2) \quad \text{(Art. VI.)}.$$

Divide out 4 and simplify.
$$\therefore\ b^2-a^2-a^2m^2=0.$$

But by (1) $b=y'-mx'$.
$$\therefore\ (y'-mx')^2-a^2-a^2m^2=0.$$

Whence
$$(x'^2-a^2)m^2-2mx'y'+(y'^2-a^2)=0 \quad . \quad . \quad (2).$$

This equation gives the gradients m_1 and m_2 of the two tangents drawn from P to the circle (compare Example 4), and since these are at right angles
$$\therefore\ m_1m_2=-1 \quad \text{(Chap. III.)}.$$

But the equation (2) gives
$$m_1m_2=\frac{y'^2-a^2}{x'^2-a^2} \quad \text{(Art. VI.)}.$$

$$\therefore \frac{y'^2 - a^2}{x'^2 - a^2} = -1.$$

$$\therefore x'^2 + y'^2 = 2a^2.$$

The locus of P is the circle $x^2+y^2=2a^2$.

RÉSUMÉ

1. The equation $x^2+y^2=a^2$ represents a circle whose centre is 0 and whose radius is a units.

Corollary.—$x^2+y^2=0$ is the equation to a point circle at the origin.

2. The expression $x^2+y^2-a^2$ is negative for points within, and positive for points without the circle $x^2+y^2=a^2$.

3. To obtain the intersections of a line and a circle, solve their equations.

4. If the solutions are coincident the line is a tangent.

5. $x=a\cos\theta$ and $y=a\sin\theta$ are the polar co-ordinates of a point P on the circle $x^2+y^2=a^2$, where $\theta=X\hat{O}P$. This angle is called the vectorial angle of the point P.

6. If x_1 and x_2 are the roots of the quadratic equation

$$ax^2+bx+c=0,$$

then (1) $x_1+x_2=-\dfrac{b}{a}$,

(2) $x_1x_2=\dfrac{c}{a}$.

If the roots are equal, then

(3) $b^2=4ac$.

EXAMPLES

1. Write down the equations to the circles whose centres are the origin and radii 1, 3, and 7 units respectively.

2. What is the length of the radius of the circle $x^2+y^2=16$?

3. Draw the circle whose equation is $x^2+y^2=13$.

4. Calculate to two places of decimals the radius of the circle $5x^2+5y^2=22$.

5. AB is a fixed straight line 6 units in length. P is a variable point such that $PA^2+PB^2=AB^2$. Find its locus.

6. A variable straight line cuts the axes at P and Q so that PQ is always 10 units in length. Find the locus of the mid-point of PQ. (See Art. VII. Ex. 2.)

7. Where does the straight line $y=2x+1$ cut the circle $x^2+y^2=10$? Verify the results graphically.

8. Prove that the line $y=3x-10$ touches the circle of last example at the point $(3, -1)$. Show this on squared paper.

9. Prove that the line $2x+3y=13$ touches the circle $x^2+y^2=13$ at the point $(2, 3)$. Show this graphically.

Prove also that the line through the centre perpendicular to the given one passes through the point of contact.

10. Find the mid-point of the chord cut off from the line $x=2y-4$ by the circle $x^2+y^2=15$.

11. Find the mid-point of the chord cut off by the circle $x^2+y^2=36$ from the line $5x+2y=10$. Verify the result by graphs.

12. Prove that the line $4x+3y=24$ does not cut the circle $x^2+y^2=4$. Draw the graphs.

13. The straight line $x=2y+c$ touches the circle $x^2+y^2=5$. What values can c have?

14. Find the slopes of the tangents drawn from the point $(0, 4)$ to the circle $x^2+y^2=6$.

15. Prove that two tangents can be drawn from a point P to a circle.

(*Hint.*—Take the diameter through P as axis of y.)

16. A and B are fixed points 10 units apart. P is a variable point such that PA is perpendicular to PB. Find the locus of P. (See Ex. 3 Art. I.)

17. A circle of variable radius and given centre O cuts the axes at P and Q. Prove that the locus of the mid-point of PQ is a line through O.

18. A circle having the origin as centre touches the line $x-3y+6=0$. Find its radius. (Art. VII. Ex. 3.)

19. Find the radii of the circles whose centres are at O and which touch (i.) the line $y=mx+b$ and (ii.) the line $lx+my+n=0$.

20. From a variable point P perpendiculars PN and PM are drawn to the axes. If the diagonal MN of the rectangle $ONPM$ is always 4 units in length, prove that the locus of P is a circle with O as centre.

21. A point moves so that the square of the sum of its co-ordinates is equal to a given square plus twice the rectangle contained by its co-ordinates. Find its locus.

22. A point moves so that the square of the difference of its co-ordinates plus twice the rectangle contained by them is equal to a given square. Find its locus.

23. A variable straight line cuts the axes at P and Q so that PQ is always 6 units in length. If the area of $\Delta\ OPQ$ is 2 sq. units, prove that the locus of the foot of the perpendicular from O to PQ is a circle.

(*Hint.*—Work out the length of the perpendicular from O to PQ.)

24. P is a point on the circle $x^2+y^2=9$ such that $X\hat{O}P=48°$. Express the co-ordinates of P in polar form. (Art. V.)

25. Show that the point (3, 4) lies on the circle $x^2+y^2=25$, and express the co-ordinates in polar form.

26. Show that the point $(-15, 8)$ lies on the circle $x^2+y^2=289$, and express the co-ordinates in polar form.

27. Prove that the point $(a \cos \theta,\ a \sin \theta)$ lies on the circle $x^2+y^2=a^2$ for all values of θ.

28. P is any point on the circumference of a circle whose diameter is AB. Q and R are the mid-points of PA and PB respectively. Find the locus of S, the mid-point of QR, taking $AB=2a$ and expressing the co-ordinates of P in polar form.

29. Any diameter of the circle $x^2+y^2=a^2$ is taken and from its ends perpendiculars are drawn to the line $lx+my+n=0$. Prove that their sum is constant. (Art. V. Ex. 2.)

30. Find the locus of the mid-points of a system of parallel chords of the circle $x^2+y^2=5$, the slope being 135°. (Art. VII. Ex. 6.)

31. A system of chords of the circle $x^2+y^2=8$ has a gradient of $\frac{3}{5}$. Find the locus of their mid-points.

32. From the centre of the circle $x^2+y^2=6$ a perpendicular is drawn to the chord $x+3y=10$. Prove that the chord is bisected. (Art. VII. Ex. 7.)

33. The mid-point of the chord cut off from the line $3x+5y=6$ by the circle $x^2+y^2=4$ is joined to the centre. Prove that the join is perpendicular to the chord.

34. Find the locus of the intersections of tangents to the circle $x^2+y^2=25$ which are at right angles to one another. (Art. VII. Ex. 8.)

35. Prove that if the straight line $y=mx+b$ touches the circle $x^2+y^2=a^2$ then its equation can be written in the form

$$y=mx\pm a\sqrt{1+m^2}.$$

Use the result to find the slopes of the tangents drawn from the point (3, 5) to the circle $x^2+y^2=4$. Verify the results by an accurate figure. (Art. VII. Ex. 5.)

36. AB is a given straight line whose mid-point is O. N is taken on OA so that $ON = \frac{3}{5}OA$. A semicircle is described on AB having C as its highest point. A perpendicular is drawn to AB at N and meets BC at P. Prove that the circumference bisects NP.

37. A and B are two fixed points. A variable point P moves so that $PA^2 + PB^2$ is constant. Prove that the locus of P is a circle.

38. Through a given point B any two chords of a circle, at right angles to one another, are drawn. P and Q are the mid-points of the chords. Prove that the mid-point of PQ is fixed.

(*Hint.*—Take B on the y-axis.)

39. The point $(1, -4)$ lies on a circle having O as centre. What is the equation to the circle ?

40. The point (x_1, y_1) lies on a circle whose centre is the origin. What is the equation to the circle ?

41. Are the following points within or without the circle $x^2 + y^2 = 30$?—

$(1, 1), (2, 3), (-5, 6), (-4, -4), (-3, 1), (2, -6)$.

42. Find the length of the tangents from

(1) the point $(4, 3)$ to the circle $x^2 + y^2 = 5$.
(2) the point $(6, 4)$ to the circle $x^2 + y^2 = 9$,
(3) the point $(0, -7)$ to the circle $x^2 + y^2 = 16$,
(4) the point $(-5, 8)$ to the circle $x^2 + y^2 = 40$.

43. A square $ABCD$ is inscribed in a circle.
P is any point on the circumference.
Prove that $PA^2 + PB^2 + PC^2 + PD^2 =$ twice the square on the diameter.

44. The graph of $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x-axis at P and Q and the y-axis at R and S. Prove by the help of Art. VI. Prop. 2, that P, Q, R and S are concyclic.

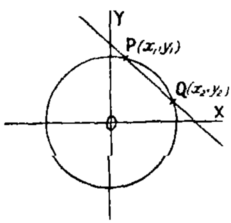
45. $ABCD$ is a square. From a variable point P perpendiculars PK, PL, and PM are drawn to BC, CD, and DB respectively. If $PM^2 = PK \cdot PL$ prove that the locus of P is a circle with centre A and radius AB.

(*Hints.*—Obtain the equation to BD from "Intercept Form." Let $P = (x', y')$ and express PK and PL in terms of x', y' and the side of the square.)

CHAPTER VI

CHORDS, TANGENTS, AND NORMALS TO A CIRCLE

I. *Find the gradient of the line joining two points* (x_1, y_1) *and* (x_2, y_2) *on the circle* $x^2+y^2=a^2$.



Let $P=(x_1, y_1)$ and $Q=(x_2, y_2)$.

Then the gradient of PQ is

$$\frac{y_1-y_2}{x_1-x_2} \quad \text{(Chap. III.)}.$$

But since P and Q lie on the circle,

$$\therefore x_1^2+y_1^2=a^2,$$

and $$x_2^2+y_2^2=a^2.$$

Subtract . . $(x_1^2-x_2^2)+(y_1^2-y_2^2)=0$,

$$\therefore (y_1-y_2)(y_1+y_2)=-(x_1-x_2)(x_1+x_2),$$

$$\therefore \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}.$$

Hence the gradient of the chord PQ is $-\dfrac{x_1+x_2}{y_1+y_2}$.

Note.—The formula $\dfrac{y_1-y_2}{x_1-x_2}$ also gives the gradient, but that just found has a special advantage when dealing with the tangent, as we shall see.

Example 1.—*Prove that the points* (7, 6) *and* (9, 2) *lie on the circle* $x^2+y^2=85$, *and find the slope of their join.*

(i.) From the equation we have

when $x=7$ then $y=\pm 6$,

and when $x=9$ then $y=\pm 2$.

Hence the points (7, 6) and (9, 2) lie on the circle.

(ii.) $$\tan\psi = -\frac{9+7}{2+6} = -2 \quad \text{(Art. I.)}$$

$$\therefore \psi = 116^\circ\ 74'.$$

EXAMPLE 2.—*Find the slope of the line joining the points on the circle $x^2+y^2=9$, whose vectorial angles are 20° and 50°.*

The points are $(3\cos 20^\circ, 3\sin 20^\circ)$ and $(3\cos 50^\circ, 3\sin 50^\circ)$,

$$\therefore \tan\psi = -\frac{3\cos 20^\circ + 3\cos 50^\circ}{3\sin 20^\circ + 3\sin 50^\circ} = -1\cdot 4281 \quad \text{(Art. I.)},$$

$$\therefore \psi = 125^\circ.$$

We shall verify this result by using the formula

$$\tan\psi = \frac{y_1-y_2}{x_1-x_2},$$

$$\therefore \tan\psi = \frac{3\sin 20^\circ - 3\sin 50^\circ}{3\cos 20^\circ - 3\sin 50^\circ} = -1\cdot 4281,$$

$$\therefore \psi = 125^\circ \text{ as before.}$$

EXAMPLE 3.—*Find the locus of the mid-points of a system of parallel chords of the circle $x^2+y^2=a^2$.* (See Chap. V. Art. VII. Ex. 6).

Let $P\equiv(x_1, y_1)$ and $Q\equiv(x_2, y_2)$ be the extremities of one chord of the system, and $R\equiv(x', y')$ be the mid-point of PQ.

Let the gradient of the system of parallels be k.

Then $$k = -\frac{x_1+x_2}{y_1+y_2}.$$

But $$x' = \frac{x_1+x_2}{2} \text{ and } y' = \frac{y_1+y_2}{2} \quad \text{(Chap. I.)},$$

$$\therefore k = -\frac{x'}{y'},$$

$$\text{or } y' = -\frac{1}{k}x'.$$

The locus of $R\equiv(x', y')$ is the straight line $y = -\frac{1}{k}x$. It passes through O and is perpendicular to the parallels, for $k \times -\frac{1}{k} = -1$.

EXAMPLE 4.—*The straight line drawn from the centre O of a circle to the mid-point of a chord PQ is perpendicular to the chord.*

Let the equation to the circle be

$$x^2+y^2=a^2.$$

Let $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$ and R the mid-point of $PQ\equiv(x', y')$.

Then $$x' = \frac{x_1+x_2}{2} \text{ and } y' = \frac{y_1+y_2}{2} \quad . \quad . \quad . \quad (1).$$

The gradient of OR is $\frac{y'}{x'}$ (Chap. III.).

The gradient of $PQ = -\frac{x_1+x_2}{y_1+y_2}$ (present article)

$$= -\frac{x'}{y'} \text{ (by 1).}$$

Hence the product of the gradients is -1.
The lines are therefore perpendicular.

II. *Equation to the tangent at the point $P \equiv (x_1, y_1)$ on the circle $x^2 + y^2 = a^2$.*

The gradient of the line which passes through two points (x_1, y_1) and (x_2, y_2) on the circle is, by last article, $-\frac{x_1+x_2}{y_1+y_2}$.

Now if the line is a tangent the points (x_1, y_1) and (x_2, y_2) are coincident, so that $x_2 = x_1$ and $y_2 = y_1$,

$$\therefore \text{ the gradient of the tangent at } (x_1, y_1) = -\frac{2x_1}{2y_1}$$

$$= -\frac{x_1}{y_1} \quad (1).$$

Hence the equation to the tangent is of the form

$$y = -\frac{x_1}{y_1}x + b \quad \text{(Chap. III. Art. VII.).}$$

But (x_1, y_1) is a point on the tangent,

$$\therefore y_1 = -\frac{x_1^2}{y_1} + b.$$

$$\therefore b = y_1 + \frac{x_1^2}{y_1} = \frac{x_1^2 + y_1^2}{y_1} = \frac{a^2}{y_1} \text{ (since } (x_1, y_1) \text{ is on the circle),}$$

$$\therefore y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1},$$

$$\therefore xx_1 + yy_1 = a^2 \quad . \quad . \quad . \quad (2).$$

This is the equation to the tangent at the point (x_1, y_1). It is easily recalled, as we have merely to express the equation to the circle in full, thus, $xx + yy = a^2$, and then replace one x by x_1 and one y by y_1.

Note.—If we had attempted to find the gradient of the tangent from the formula $\dfrac{y_1 - y_2}{x_1 - x_2}$, we would have failed, as on putting $y_2 = y_1$ and $x_2 = x_1$ the expression becomes $\dfrac{0}{0}$, which is indeterminate.

EXAMPLE 1.—*Find the slope of the tangent at the point* $(1 \cdot 6, 1 \cdot 2)$ *on the circle* $x^2 + y^2 = 4$.

First we assure ourselves that the point does lie on the circle. We have when $x = 1 \cdot 6$ and $y = 1 \cdot 2$

$$x^2 + y^2 = (1 \cdot 6)^2 + (1 \cdot 2)^2 = 4.$$

The point is therefore on the circle,

$$\therefore \tan \psi = -\frac{1 \cdot 6}{1 \cdot 2} = -1 \cdot 3333 \text{ (approx.) (by formula (1) of this article),}$$

$$\therefore \psi = 126^\circ\ 52' \text{ (nearly).}$$

EXAMPLE 2.—*Find the equation to the tangent at the point* $(2, 5)$ *on the circle* $x^2 + y^2 = 29$.

From the equation to the circle we have

$$y^2 = 25 \text{ when } x = 2,$$
$$\therefore y = \pm 5.$$

Hence the point $(2, 5)$ is on the circle.

The equation to a tangent is

$$xx_1 + yy_1 = a^2.$$

In this case $x_1 = 2$ and $y_1 = 5$ and $a^2 = 29$,

$$\therefore 2x + 5y = 29$$

is the equation required.

EXAMPLE 3.—*What is the equation of the tangent to the circle* $x^2 + y^2 = 13$ *at the point* $(3, -4)$?

We must as usual test whether the point is really on the circle.

When $\qquad x = 3$ and $y = -4$,

then $\qquad x^2 + y^2 = 9 + 16 = 25.$

Hence the point does not lie on the circle. There can therefore be no tangent at the point.

EXAMPLE 4.—*Find the equations to the tangents at the points where the straight line* $x = 5$ *cuts the circle* $x^2 + y^2 = 34$, *and prove that they intersect on the x-axis.*

When $\qquad x = 5$ then $y^2 = 9$,

$$\therefore y = \pm 3.$$

The straight line $x = 5$ therefore cuts the circle at the points $(5, 3)$ and $(5, -3)$.

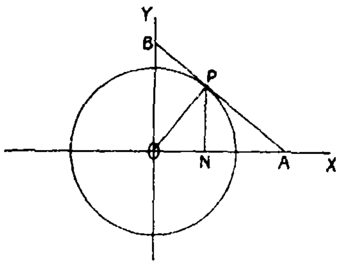
The equations of the tangents at these points are

$$5x+3y=34,$$
$$\text{and } 5x-3y=34.$$

The solutions of these equations are $x=6{\cdot}8$ and $y=0$.
Hence the tangents intersect on the x-axis at the point $(6{\cdot}8, 0)$.

EXAMPLE 5.—*The tangent at a point P on the circle $x^2+y^2=a^2$ cuts XX' at A. N is the foot of the ordinate of P.*

Prove that $OA{\cdot}ON=OP^2$.



Let $P\equiv(x_1, y_1)$.

Then the tangent at P is

$$xx_1+yy_1=a^2.$$

At A, $y=0$. The corresponding value of x, derived from the equation to the tangent is $\dfrac{a^2}{x_1}$.

$$\therefore OA=\frac{a^2}{ON},$$

$$\therefore OA{\cdot}ON=OP^2.$$

EXAMPLE 6.—*The tangent to the circle $x^2+y^2=5$ at the point P on the circumference, cuts the x-axis at A and the y-axis at B. Prove that $\dfrac{1}{OA^2}+\dfrac{1}{OB^2}=\dfrac{1}{5}$.*

Let $P\equiv(x_1, y_1)$. The tangent at P is therefore the line $xx_1+yy_1=5$.

Hence $$A\equiv\left(\frac{5}{x_1}, 0\right) \text{ and } B\equiv\left(0, \frac{5}{y_1}\right),$$

$$\therefore OA=\frac{5}{x_1} \text{ and } OB=\frac{5}{y_1},$$

$$\therefore \frac{1}{OA^2}+\frac{1}{OB^2}=\frac{x_1^2}{25}+\frac{y_1^2}{25}=\frac{x_1^2+y_1^2}{25}.$$

Now $x_1^2+y_1^2=5$ since (x_1, y_1) is a point on the circumference.

$$\therefore \frac{1}{OA^2}+\frac{1}{OB^2}=\frac{1}{5}.$$

Definition.—The perpendicular to a tangent at its point of contact with a curve is called the "Normal to the curve at the point."

III. *Equation of the normal to the circle $x^2+y^2=a^2$ at the point $P\equiv(x_1, y_1)$ on it.*

The equation to the tangent at P is

$$xx_1+yy_1=a^2.$$

Therefore $xy_1 - yx_1 = k$ is the equation to some line at right angles to the tangent (Chap. III.).

If $P \equiv (x_1, y_1)$ lies on this line, then

$$x_1y_1 - y_1x_1 = k,$$
$$\therefore k = 0.$$

Hence $xy_1 - yx_1 = 0$ or $\dfrac{x}{x_1} = \dfrac{y}{y_1}$ is the equation to the normal.

It passes through the origin.

Corollary.—The normal is the radius from the centre to the point of contact.

Example 1.—*What is the equation of the normal to the circle $x^2 + y^2 = 5$ at the point* (2, 1) ?

Having verified as usual that the point is on the circle, the equation to the normal is, by the work above,

$$\frac{x}{2} = \frac{y}{1} \text{ or } y = \cdot 5x.$$

Example 2.—*In a circle prove that the angle between two tangents is equal to the angle between the normals at their points of contact.*

Let the equation to the circle be

$$x^2 + y^2 = a^2.$$

Let the points of contact be

$$P \equiv (x_1, y_1) \text{ and } Q \equiv (x_2, y_2).$$

The gradients of the tangents at P and Q are $-\dfrac{x_1}{y_1}$ and $-\dfrac{x_2}{y_2}$ respectively. (Art. II.)

The gradients of the corresponding normals are therefore $\dfrac{y_1}{x_1}$ and $\dfrac{y_2}{x_2}$.

If θ is the angle between the tangents, then

$$\tan\theta = \frac{-\dfrac{x_1}{y_1} + \dfrac{x_2}{y_2}}{1 + \dfrac{x_1x_2}{y_1y_2}} = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2} \quad \text{(Chap. III., last example).}$$

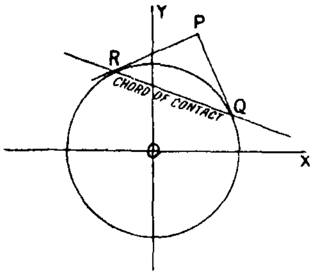
If ϕ is the angle between the normals, then

$$\tan\phi = \frac{\dfrac{y_1}{x_1} - \dfrac{y_2}{x_2}}{1 + \dfrac{y_1y_2}{x_1x_2}} = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2},$$

$$\therefore \theta = \phi.$$

IV. *The chord of contact of the tangents from an external point P to a circle.*

EXAMPLE 1.—*From the point $P \equiv (5, 3)$ tangents PQ and PR are drawn to the circle $x^2+y^2=6$. Prove that Q and R lie on the line $5x+3y=6$.*



Let $Q \equiv (x_1, y_1)$ and $R \equiv (x_2, y_2)$.

If Q and R lie on the line $5x+3y=6$, then must

$$5x_1+3y_1=6 \quad . \quad (1),$$

$$\text{and } 5x_2+3y_2=6 \quad . \quad (2).$$

Now the equation to the tangent at Q is $xx_1+yy_1=6$, and $P \equiv (5, 3)$ lies on it.

$$\therefore 5x_1+3y_1=6.$$

Hence (1) is true.

Again the equation to the tangent at R is xx_2+yy_2-6, and P lies on it also.

$$\therefore 5x_2+3y_2=6.$$

Thus (2) is true also.

Since, then, Q and R both lie on the line $5x+3y=6$, this equation is that of the chord of contact of the two tangents from P to the circle.

EXAMPLE 2.—*Find the equation to the chord of contact of the tangents from $P \equiv (4, 6)$ to the circle $x^2+y^2=3$.*

Use the figure and conventions of last example.

The tangent at Q is $xx_1+yy_1=3$.

Since $P \equiv (4, 6)$ lies on it,

$$\therefore 4x_1+6y_1=3.$$

Hence (x_1, y_1) is a point on the line

$$4x+6y=3 \quad . \quad . \quad . \quad (1).$$

The tangent at R is $xx_2+yy_2=3$.

Since $P \equiv (4, 6)$ lies on it,

$$\therefore 4x_2+6y_2=3.$$

Thus (x_2, y_2) is also a point on the line

$$4x+6y=3 \quad . \quad . \quad . \quad . \quad (2).$$

As Q and R therefore both lie on the line $4x+6y-3$, it follows that the equation is that of the chord of contact of the tangents from P to the circle.

V. *Find the equation of the chord of contact of the tangents from $P \equiv (x', y')$ to the circle $x^2+y^2=a^2$.*

Use the figure of last article, and as before let $Q \equiv (x_1, y_1)$ and $R \equiv (x_2, y_2)$.

The equation of the tangent at Q is

$$xx_1+yy_1=a^2.$$

P lies on this tangent,

$$\therefore\ x'x_1+y'y_1=a^2.$$

Hence (x_1, y_1) is a point on the line

$$x'x+y'y=a^2 \quad . \quad . \quad . \quad (1).$$

The equation of the tangent at R is

$$xx_2+yy_2=a^2.$$

P lies on it,

$$x'x_2+y'y_2=a^2$$

Thus R is also a point on the line

$$x'x+y'y=a^2 \quad . \quad . \quad . \quad (2).$$

It follows that $xx'+yy'=a^2$ {see (1) and (2)} is the equation to the line which passes through the points Q and R. It is the equation of the chord of contact of the tangents from P to the circle.

The similarity of this equation to that of the tangent at a point on the same circle has to be remarked. This might be expected, since the angle between the tangents becomes nearer and nearer a straight one as P approaches the circle. When, ultimately, P lies on the circle the tangents are in the same straight line, and coincide with the chord of contact

Whether the equation is that of the chord of contact of two tangents, or that of the tangent at a point on the circle, is easily determined

If $xx'+yy'=a^2$ is the equation to a tangent, then must (x', y') satisfy the equation to the circle on which it lies, that is to say, we must have $x'^2+y'^2=a^2$.

If the equation is that of the chord of contact of tangents from (x', y') to the circle, then since (x', y') does not lie on the circle the co-ordinates cannot satisfy the equation of the circle.

The process of writing down the equation to a chord of contact is the same as for a tangent.

EXAMPLE 1.—*Write down the equation to the chord of contact of tangents from the point* $(2, -5)$ *to the circle* $x^2+y^2=7$.

Here $x'=2$ and $y'=-5$.

The required equation is therefore

$$2x-5y=7.$$

EXAMPLE 2.—*Find the points of contact of the tangents from the point* $(-3, 4)$ *to the circle* $x^2+y^2=2$.

We shall attain our end if we find where the chord of contact cuts the circumference.

The equation to the chord of contact is

$$-3x+4y=2 \quad . \quad . \quad . \quad . \quad (1).$$

The equation to the circle is

$$x^2+y^2=2 \quad . \quad . \quad . \quad . \quad (2).$$

If we solve equations (1) and (2) in the usual way, we have the following solutions:

x	$\cdot 85$	$-1\cdot 33$
y	$1\cdot 4$	$-\cdot 50$

EXAMPLE 3.—*The chord of contact of tangents from* P *to a circle is perpendicular to the line joining* P *to the centre.*

Let the equation to the circle be

$$x^2+y^2=a^2$$

and $P\equiv(x', y')$.

The equation to the chord of contact of tangents from P is

$$xx'+yy'=a^2.$$

The gradient of the chord is therefore

$$-\frac{x'}{y'} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The gradient of OP is $\frac{y'}{x'}$.

Since the product of their gradients is -1, the two lines are perpendicular.

EXAMPLE 4.—*Find the point of intersection of the tangents drawn at the extremities of the chord intercepted by the circle* $x^2+y^2=9$ *on the line* $2x+5y=10$.

Let (x', y') be the point of intersection of the tangents.

Their chord of contact is $xx'+yy'=9$.

This line must therefore be identical with $2x+5y=10$.

The respective intercepts of these lines on the two axes must be the same.

$$\therefore \frac{9}{x'}=5 \text{ (x-intercepts) and } \frac{9}{y'}=2 \text{ (y-intercepts)},$$

$$\therefore x'=1\cdot 8 \text{ and } y'=4.5,$$

$$\therefore (x', y')\equiv(1\cdot 8, 4\cdot 5).$$

VI. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Find the locus of the intersection of tangents to a circle, which are at right angles to one another.*

Let the equation to the circle be

$$x^2+y^2=a^2.$$

Then $xx_1+yy_1=a^2$ and $xy_1-yx_1=a^2$ are the equations to two tangents, the first at the point (x_1, y_1), and the second at the point $(y_1, -x_1)$. They are at right angles, since the coefficients of x and y are interchanged in the equations, and the sign of one of them reversed (Chap. III.).

Let (x', y') be the point of intersection of the tangents.

$$\therefore x'x_1+y'y_1=a^2 \quad . \quad . \quad . \quad (1),$$
$$\text{and } x'y_1-y'x_1=a^2 \quad . \quad . \quad (2).$$

Square both equations.

$$\therefore x'^2x_1^2+y'^2y_1^2+2x'y'x_1y_1=a^4,$$
$$\text{and } x'^2y_1^2+y'^2x_1^2-2x'y'x_1y_1=a^4.$$

Add these results.

$$\therefore x'^2(x_1^2+y_1^2)+y'^2(x_1^2+y_1^2)=2a^4 \quad . \quad (3).$$

But $x_1^2+y_1^2=a^2$ since (x_1, y_1) is on the circle.

Hence on dividing out a^2 from equation (3) we have

$$x'^2+y'^2=2a^2.$$

The locus of (x', y') is the circle whose equation is

$$x^2+y^2=2a^2. \quad \text{(See Chap. V. Art. VII. Ex. 8).}$$

EXAMPLE 2.—*P is a point on a circle. Through the centre a line is drawn parallel to the tangent at P and cutting the circumference at Q. Prove that the tangent at Q is perpendicular to the tangent at P.*

Let the equation to the circle be

$$x^2+y^2=a^2.$$

Let $P\equiv(x_1, y_1)$ and $Q\equiv(x_2, y_2)$.

The tangent at P is the line

$$xx_1+yy_1=a^2.$$

The line through the origin parallel to this is

$$xx_1+yy_1=0 \quad \text{(Chap. III.)}.$$

Since $Q\equiv(x_2, y_2)$ lies on this line,

$$\therefore x_2x_1+y_2y_1=0 \quad . \quad . \quad . \quad . \quad (1).$$

Now the tangent at Q is $xx_2+yy_2=a^2$.

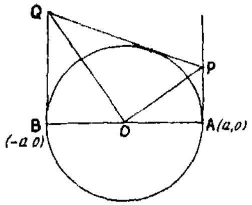
It is perpendicular to the tangent at P if

$$x_1x_2+y_1y_2=0 \quad \text{(Chap. III.)}.$$

This last result is true by (1). The tangents at P and Q are therefore perpendicular.

EXAMPLE 3.—*AB is a diameter of a circle. Tangents are drawn at A and B. Any other tangent cuts these at P and Q. Prove that PQ subtends a right angle at the centre of the circle.*

Let the equation to the circle be



$$x^2+y^2=a^2,$$

and that of PQ

$$xx_1+yy_1=a^2.$$

At P $x=a$, $\therefore$ $ax_1+yy_1=a^2$.

$$\therefore y=\frac{a^2-ax_1}{y_1},$$

$$\therefore P\equiv\left(a, \frac{a^2-ax_1}{y_1}\right).$$

Similarly $\qquad Q\equiv\left(-a, \frac{a^2+ax_1}{y_1}\right).$

The gradient of OP $=\dfrac{a^2-ax_1}{ay_1}=\dfrac{a-x_1}{y_1}$ (Chap. III.).

The gradient of OQ is $\dfrac{a^2+ax_1}{-ay_1}=-\dfrac{a+x_1}{y_1}$

Product of the gradients

$$=-\frac{a+x_1}{y_1}\times\frac{a-x_1}{y_1}$$

$$=-\frac{a^2-x_1^2}{y_1^2}.$$

Now $x_1^2+y_1^2=a^2$, since (x_1, y_1) is the point of contact of the tangent.

$$\therefore y_1^2=a^2-x_1^2.$$

Hence the product of the gradients is -1.

OP and OQ are therefore perpendicular.

EXAMPLE 4.—*Tangents to a system of concentric circles are drawn parallel to a given line. Find the locus of the points of contact.*

Take the common centre as origin, and let the equation of the given line be

$$lx+my+n=0.$$

Let the radius of one of the circles be r.

Its equation will therefore be

$$x^2+y^2=r^2.$$

Let (x_1, y_1) be the point of contact of the tangent parallel to the given line.

The equation to the tangent is

$$xx_1+yy_1=r^2.$$

It is parallel to the line $lx+my+n=0$ if

$$\frac{x_1}{l}=\frac{y_1}{m} \quad \text{(Chap. III).}$$

The locus of (x_1, y_1) is therefore the line $\dfrac{x}{l}=\dfrac{y}{m}$.

It passes through the origin and is perpendicular to $lx+my+n=0$.

EXAMPLE 5.—*The straight line* $lx+my+n=0$ *cuts the circle* $x^2+y^2=a^2$ *at* A *and* B. *Tangents are drawn at* A *and* B. *Find* P *their point of intersection.*

Let $P \equiv (x', y')$.

The equation to the chord of contact of tangents from P is

$$xx'+yy'=a^2.$$

This line must therefore be identical with $lx+my+n=0$.

The lines will therefore make the same intercepts on the axes.

$$\therefore \frac{a^2}{x'} = -\frac{n}{l} \text{ (}x\text{-intercepts)},$$

$$\text{and } \frac{a^2}{y'} = -\frac{n}{m} \text{ (}y\text{-intercepts)}.$$

$$\therefore x' = -\frac{a^2 l}{n} \text{ and } y' = -\frac{a^2 m}{n},$$

$$\therefore P \equiv (x', y') \equiv \left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right).$$

EXAMPLE 6.—*A variable straight line is drawn through a fixed point* (h, k) *and cuts the circle* $x^2+y^2=a^2$ *at* P *and* Q. *Find the locus of* R, *the intersection of the tangents at* P *and* Q.

Let $R \equiv (x', y')$.

The chord of contact of tangents from R is $xx'+yy'=a^2$.

Since this line passes through (h, k).

$$\therefore hx'+ky'=a^2.$$

The locus of $R \equiv (x', y')$ is the straight line $hx+ky=a^2$.

It is the chord of contact of tangents from (h, k) to the circle.

RÉSUMÉ

1. If (x_1, y_1) and (x_2, y_2) be two points on the circle $x^2+y^2=a^2$, then the gradient of their join is $-\dfrac{x_1+x_2}{y_1+y_2}$.

2. The equation to the tangent at the point (x_1, y_1) is

$$xx_1+yy_1=a^2$$

3. The equation to the normal at the point (x_1, y_1) is

$$xy_1-yx_1=0.$$

4. The equation to the chord of contact of tangents drawn from a point (x', y') to the circle is

$$xx'+yy'=a^2.$$

EXAMPLES

1. Prove that the point (5, 2) lies on the circle $x^2+y^2=29$, and write down the equation to the tangent at the point of contact.

2. Write down the equation to the tangent at

(i.) the point (3, 4) on the circle $x^2+y^2=25$,
(ii.) the point $(-2, 6)$ on the circle $x^2+y^2=40$,
(iii.) the point $(-4, -5)$ on the circle $x^2+y^2=41$,
(iv.) the point $(1, -7)$ on the circle $x^2+y^2=50$.

Draw a squared-paper figure in the case of (i.).

3. A circle with centre O passes through the point $(-1, 3)$. The tangent thereat cuts the axes at A and B. Find the area of $\Delta\, OAB$.

4. What is the slope of the tangent at the point $(-3, 6)$ on the circle $x^2+y^2=45$?

What is the slope of the normal at the same point?

5. What is the equation of the straight line through the origin which is parallel to the tangent at the point $(2, -9)$ on the circle $x^2+y^2=85$?

6. Find the gradient of the tangent to the circle $x^2+y^2=13$ at the point (2, 3) from first principles, as in Articles I. and II.

7. Write down the equation to the normal at the point (3, 8) on the circle $x^2+y^2=73$.

8. Find the locus of the mid-points of chords of the circle $x^2+y^2=8$ which are parallel to the line $2x+5y=0$. (Art. I. Ex. 3.)

9. (6, 7) and $(-9, 2)$ are points on the circle $x^2+y^2=85$. Work out the gradient of the chord joining them by both the gradient formulae. (Art. I.)

10. In last example prove that the line from the centre to the mid-point of the chord is perpendicular to the chord.

11. A straight line cuts the axes at A and B. A circle having O as centre touches AB at P. PN is perpendicular to XX'.

If $OA \cdot ON=12$, find the equation to the circle. (Art. II. Ex. 5.)

12. AB is a diameter of a circle. From A and B perpendiculars AP and BQ are drawn to any tangent. Prove that $AP+BQ=AB$.

13. Write down the equation of the chord of contact of tangents from

(i.) the point (5, 7) to the circle $x^2+y^2=2$,
(ii.) the point $(8, -3)$ to the circle $x^2+y^2=4$,
(iii.) the point $(-4, -6)$ to the circle $x^2+y^2=5$.

Work out the equation of (i.) *ab initio*, as in Art. IV. Ex. 2.

14. From the point $P\equiv(3, 8)$ tangents are drawn to the circle $x^2+y^2=6$.

(i.) Prove OP perpendicular to the chord of contact.
(ii.) If OP cuts the chord of contact at Q, calculate OQ.
(iii.) Prove $OP \cdot OQ = 6$.

15. The straight line $2x+3y=6$ cuts the circle $x^2+y^2=25$ at the points P and Q. Find the point of intersection of the tangents at P and Q. (Art. V. Ex. 4.)

16. Find the point of intersection of the tangents at the extremities of the chord cut off by the circle $x^2+y^2=36$ from the line $x=3y+6$.

17. Find the points of contact of the tangents drawn from the point $(5, -4)$ to the circle $x^2+y^2=3$. (Art. V. Ex. 2.)

18. Find the points of contact of the tangents drawn from the point $(10, 6)$ to the circle $x^2+y^2=8$.

19. A tangent to the circle $x^2+y^2=10$ cuts the axes at A and B.

Prove $\dfrac{1}{OA^2}+\dfrac{1}{OB^2}=\dfrac{1}{10}$.

20. Tangents are drawn from a point (x', y') to the circle $x^2+y^2=a^2$. OP cuts the chord of contact at Q.
Prove (i.) OQ perpendicular to the chord of contact.
(ii.) $OP \cdot OQ=a^2$.

21. A variable straight line is drawn through the point $(0, 3)$ and cuts the circle $x^2+y^2=12$ at P and Q. Prove that the locus of the intersection of the tangents at P and Q is a line parallel to XX'.
(*Hint.*—State the condition that the chord of contact of (x', y') passes through $(0, 3)$. See Art. VI. Ex. 6.)

22. A tangent is drawn through the point $(3, -1)$ to the circle $x^2+y^2=a^2$. If its gradient is $\frac{1}{2}$, find its point of contact.

23. The tangent to $x^2+y^2=a^2$ at the point (x_1, y_1) cuts the axes at A and B. Prove that $\Delta OAB=\dfrac{a^4}{2x_1y_1}$.

24. Tangents PA and PB are drawn to a circle. Prove analytically that the length of the perpendicular from A to PB is equal to the length of the perpendicular from B to PA.

25. Prove that if the tangents at two points on a circle are perpendicular, then the lines joining the centre to the points are perpendicular also.

26. Prove that the tangents to the circle $x^2+y^2=a^2$ at the extremities of the chord $lx+my=1$ intersect at the point (la^2, ma^2). (Art. VI. Ex. 5.)
Hence show that the locus of the intersections of tangents at the extremities of a system of parallel chords of a circle is the diameter perpendicular to the system.

27. Chords of a circle are drawn through a given point. Prove

that the tangents at their extremities intersect on a fixed line. (Art. VI. Ex. 6.)

28. AB is a diameter of a circle. Tangents are drawn at A and B. These are cut by any other tangent at P and Q. Prove that PQ subtends a right angle at the centre. (Art. VI. Ex. 3.)

29. If θ_1 and θ_2 be the vectorial angles of two points on the circle $x^2+y^2=a^2$, prove that their join is perpendicular to the line whose slope is $\frac{\theta_1+\theta_2}{2}$.

30. AB is a diameter of a circle. A line through B cuts the circle at P and the tangent at A in Q. Prove that the tangent at P bisects AQ.

(*Hints.*—Take AB as axis of y, and the equation to the line through B in gradient form.)

31. On the circle $x^2+y^2=a^2$ the point $Q \equiv (y_1, x_1)$ is taken. QN and QM are drawn perpendicular to the axes. If P is a point on the tangent at (x_1, y_1), prove that the square on the radius is double the area of the quadrilateral $ONPM$.

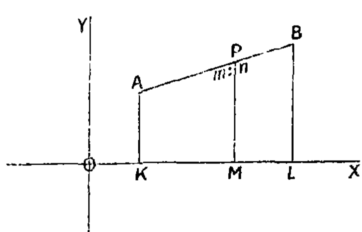
CHAPTER VII

MORE ABOUT CO-ORDINATES: THE FORMULAE $x' = \dfrac{mx_1 + nx_2}{m+n}$, $y' = \dfrac{my_2 + ny_1}{m+n}$: THE FORMULAE $x_1 = x_2 + r \cos \psi$, $y_1 = y_2 + r \sin \psi$: HARMONIC DIVISION

I. Before dealing with the subjects of this chapter we again remind the reader of the importance of what we called the "sense" of a line. (Chap. I.)

The sense of a line is the distinction between *LB* and *BL*, as, for example, the journeys London to Bristol and Bristol to London. The differentiation is accomplished by marking *LB* + and *BL* −, or *vice versa*, according to the convenience of the worker, but once the convention has been stated it must be rigorously observed.

II. *A point P is taken on the line through two given points $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ so that $AP : PB = m : n$.* ***Find the*** *co-ordinates of P.*



Let $P \equiv (x', y')$.

Draw *AK*, *BL*, and *PM* perpendicular to *XX'*.

Then by ordinary geometry

$$\frac{KM}{ML} = \frac{AP}{PB} = \frac{m}{n},$$

$$\therefore \frac{x' - x_1}{x_2 - x'} = \frac{m}{n},$$
$$\therefore nx' - nx_1 = mx_2 - mx',$$
$$\therefore (m+n)x' = mx_2 + nx_1,$$
$$\therefore x' = \frac{mx_2 + nx_1}{m+n}.$$

Similarly, by drawing perpendiculars to the y-axis it can be shown that

$$y' = \frac{my_2 + ny_1}{m+n}.$$

Note.—These results are easily written down according to the following scheme.

$$\begin{array}{ccc} m & & n \\ & \times & \\ x_1 & & x_2 \end{array} \qquad \begin{array}{ccc} m & & n \\ & \times & \\ y_1 & & y_2 \end{array}$$

The products in the numerators of the formulae follow the cross lines.

Corollary 1.—If the point P does not lie between A and B, but is external to them, then the reader can easily draw the necessary figure and verify that

$$x' = \frac{mx_2 - nx_1}{m-n}$$

$$\text{and } y' = \frac{my_2 - ny_1}{m-n}$$

There is, however, no need to deal with this case separately, for we have only to note that the segments AP and PB of the line now have opposite senses, so that $\frac{AP}{PB} = \frac{m}{-n}$ and we have merely to replace n by $-n$ in the formulae.

N.B.—We shall now use λ, the Greek letter for l called "lambda."

Corollary 2.—If $\frac{AP}{PB} = \lambda \left(i.e.\ \frac{\lambda}{1}\right)$, then the formulae modify to

$$\frac{\lambda x_2 + x_1}{\lambda + 1} \text{ and } \frac{\lambda y_2 + y_1}{\lambda + 1},$$

that is, $$x' = \frac{x_1 + \lambda x_2}{1 + \lambda} \text{ and } y' = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

The distinction between $\frac{m}{n}$ and λ is merely that between $\frac{4}{5}$ and $\cdot 8$; that is to say, we may write

$$\frac{AP}{PB} = \tfrac{4}{5} \text{ or } \frac{AP}{PB} = \cdot 8$$

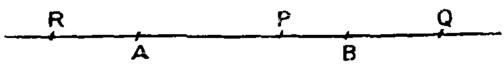
according to our convenience.

Corollary 3.—If P is the mid-point of AB, then $\frac{m}{n}$ (or λ) $= 1$ and the formulae become

$$\frac{x_1 + x_2}{2} \text{ and } \frac{y_1 + y_2}{2}$$

We have already found in Chapter I. the co-ordinates of the mid-point of AB

Definition of the external and internal division of a line.—A line AB is said to be divided internally by a point P when P lies between A and B. It is said to be divided externally at P when P is on the part of the line not between A and B.



In the figure P divides AB internally, while Q and R divide it externally.

EXAMPLE 1.—*Find the co-ordinates of the point which divides the join of* (1, 2) *and* (5, 6) *internally in the ratio* 3 : 4.

$$x' = \frac{(3 \times 5) + (4 \times 1)}{3 + 4} = \frac{19}{7},$$

$$y' = \frac{(3 \times 6) + (4 \times 2)}{3 + 4} = \frac{26}{7}.$$

EXAMPLE 2.—*Find the co-ordinates of the point which divides the join of* (1, 2) *and* (5, 6) *externally in the ratio* 3 : 4.

As the segments AP and PB are now oppositely directed, we must write $\frac{3}{-4}$.

$$\therefore x' = \frac{(3 \times 5) + (-4 \times 1)}{3 - 4} = -11,$$

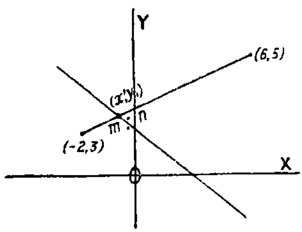
$$\text{and } y' = \frac{(3 \times 6) + (-4 \times 2)}{3 - 4} = -10.$$

The same results are obtained if we write the ratio as $\frac{-3}{4}$.

It is merely a question of agreement as to whether AP or PB is to be regarded as $+$. (See Art. I.)

EXAMPLE 3.—*In what ratio is the line joining the points* $(-2, 3)$ *and* $(6, 5)$ *divided by the line* $3x+4y=12$?

Suppose it is divided at (x', y') in the ratio (m, n).



$$\therefore x' = \frac{6m-2n}{m+n} \text{ and } y' = \frac{5m+3n}{m+n}.$$

Now (x', y') is a point on the line $3x+4y=12$.

$$\therefore \frac{3(6m-2n)}{m+n} + \frac{4(5m+3n)}{m+n} = 12.$$

Whence $26m = 6n$,

$$\therefore \frac{m}{n} = \frac{3}{13}.$$

The line is divided internally at the point (x', y') in the ratio $\frac{3}{13}$.

EXAMPLE 4.—*Find where the line joining the points* $(4, 2)$ *and* $(10, 3)$ *is intersected by the line* $2x-5y+4=0$.

Let the line joining the two points be divided at (x', y') in the ratio $\lambda : 1$ by $2x-5y+4=0$.

$$\therefore x' = \frac{4+10\lambda}{1+\lambda} \text{ and } y' = \frac{2+3\lambda}{1+\lambda}.$$

But $2x'-5y'+4=0$,

$$\therefore \frac{8+20\lambda}{1+\lambda} - \frac{10+15\lambda}{1+\lambda} + 4 = 0.$$

Whence $9\lambda = -2$,

$$\therefore \lambda = -\tfrac{2}{9}.$$

The join of the two points is divided externally at (x', y') as $2 : 9$.

Since $\lambda = -\tfrac{2}{9}$,

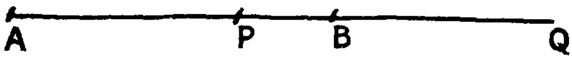
$$\therefore x' = \frac{4-\frac{20}{9}}{1-\frac{2}{9}} = \frac{16}{7}$$

$$\text{and } y' = \frac{2-\frac{6}{9}}{1-\frac{2}{9}} = \frac{12}{7}.$$

The point of intersection is $(\frac{16}{7}, \frac{12}{7})$.

III. Definition.—If the straight line joining two points A and B is divided internally at P and externally at Q in the same ratio it is divided "harmonically." P and Q are called "harmonic

conjugates" with respect to A and B, and are said to separate A and B harmonically.



Thus if P and Q harmonically separate A and B, we have

$$\frac{AP}{PB} = -\frac{AQ}{QB}.$$

(AQ and QB are oppositely directed, hence the negative sign.)

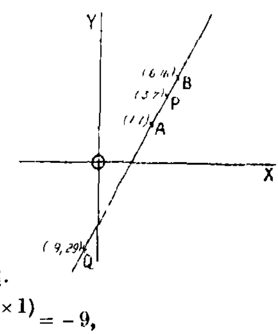
Corollary—*If P and Q harmonically separate A and B, then A and B harmonically separate P and Q.*

For since $$\frac{AP}{PB} = -\frac{AQ}{QB},$$

$$\therefore \frac{QB}{-PB} = \frac{AQ}{AP},$$

$$\text{or } \frac{QB}{BP} = -\frac{QA}{AP}.$$

EXAMPLE 1.—*$A \equiv (1, 1)$, $P \equiv (3, 7)$, and $B \equiv (6, 16)$ are points lying on the line $y = 3x - 2$. Find Q the harmonic conjugate of P with respect to A and B.*



Let $Q \equiv (x', y')$.

Let P divide AB in the ratio $\frac{m}{n}$.

Then Q divides AB in the ratio $-\frac{m}{n}$.

Since $P \equiv (3, 7)$,

$$\therefore 3 = \frac{6m + n}{m + n},$$

which gives $$\frac{m}{n} = \frac{2}{3}.$$

Hence Q divides AB in the ratio $-\frac{2}{3}$.

$$\therefore x' = \frac{(2 \times 6) + (-3 \times 1)}{2 - 3} = -9,$$

$$y' = \frac{(2 \times 16) + (-3 \times 1)}{2 - 3} = -29.$$

$$\therefore Q \equiv (-9, -29).$$

Example 2.—*The straight line $3x-4y+12=0$ cuts the axes at A and B. $P\equiv(-1, 2\cdot25)$ is taken on the line. Find Q its harmonic conjugate with respect to A and B.*

Let $Q\equiv(x', y')$.

Let P divide AB in the ratio $\frac{m}{n}$.

Then Q divides AB in the ratio $-\frac{m}{n}$.

Since $A\equiv(-4, 0)$ and $B\equiv(0, 3)$, we have, for the abscissa of P,

$$-1=\frac{-4n}{m+n},$$

which gives
$$\frac{m}{n}=\frac{3}{1}.$$

Hence Q divides AB in the ratio $-\frac{3}{1}$.

$$\therefore x'=\frac{(3\times0)+(-4\times-1)}{3-1}=2,$$

$$\text{and } y'=\frac{(3\times3)+(-1\times0)}{3-1}=\frac{9}{2}.$$

$$\therefore Q\equiv(2, 4\cdot5).$$

IV. **We shall now have occasion to use the following additional properties of quadratic equations.**

(i.) If the roots of the equation $ax^2+bx+c=0$ are equal in value but opposite in sign, then $b=0$.

For if $b=0$ the equation reduces to $ax^2+c=0$, whence

$$x=\pm\sqrt{-\frac{c}{a}}.$$

(ii.) If $a=0$, then one root of the equation is infinite.

If, in addition, $b=0$, then both roots are infinite.

Let $x=\frac{1}{y}$.

$$\therefore \frac{a}{y^2}+\frac{b}{y}+c=0.$$

$$\therefore cy^2+by+a=0.$$

One root of this equation is zero when $a=0$, and both roots are zero if, in addition, $b=0$.

But since $x=\frac{1}{y}$, we see that when $y=0$, $x=\infty$, so that if the

roots of $cy^2+by+a=0$ are zero, then those of $ax^2+bx+c=0$ are infinite.

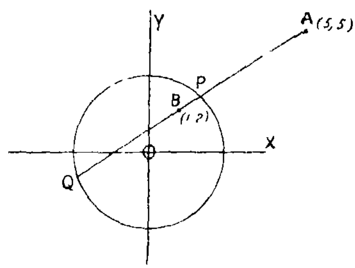
Hence one root of the equation $ax^2+bx+c=0$ is infinite if $a=0$, and both roots are infinite when $a=b=0$.

V. *Intersections of the straight line joining two points, and a circle whose centre is the origin*

EXAMPLE 1. *In what ratio does the circle $x^2+y^2=9$ divide the join of the points* $(1, 2)$ *and* $(5, 5)$?

Let $A\equiv(5, 5)$ and $B\equiv(1, 2)$

Let the circle divide the straight line AB at a point (x', y') in the ratio $\frac{\lambda}{1}$.



$$\therefore\ x'=\frac{5+\lambda}{1+\lambda} \text{ and } y'=\frac{5+2\lambda}{1+\lambda}.$$

Now (x', y') is on the circle $x^2+y^2=9$,

$$\therefore \left(\frac{5+\lambda}{1+\lambda}\right)^2+\left(\frac{5+2\lambda}{1+\lambda}\right)^2=9,$$

$$\therefore (5+\lambda)^2+(5+2\lambda)^2=9(1+\lambda)^2,$$

$$\therefore 4\lambda^2-12\lambda-41=0$$

$$\therefore \lambda=5\cdot035, \text{ or } -2\cdot035 \text{ (approx.)}.$$

There are therefore two values of λ, corresponding to the two points P and Q, in which AB cuts the circle, and we note that they are opposite in sign, showing that one point is within and the other without the circle.

If λ_1 and λ_2 are the roots of the quadratic, then

$$\frac{AP}{PB}=\lambda_1 \text{ and } \frac{AQ}{QB}=\lambda_2$$

Very nearly, $$\frac{AP}{PB}=5 \text{ and } \frac{AQ}{QB}=-2.$$

EXAMPLE 2.—*In what ratio does the circle $x^2+y^2=4$ divide the join of the points* $(1\cdot2, 1\cdot6)$ *and* $(5, -\cdot6)$?

(i.) Let the circle divide the join in the ratio $(\lambda : 1)$ at the point (x', y').

Then $$x'=\frac{1\cdot2+5\lambda}{1+\lambda} \text{ and } y'=\frac{1\cdot6-\cdot6\lambda}{1+\lambda}.$$

Now $x'^2+y'^2=4$, since (x', y') is on the circle,

$$\therefore \left(\frac{1\cdot2+5\lambda}{1+\lambda}\right)^2+\left(\frac{1\cdot6-\cdot6\lambda}{1+\lambda}\right)^2=4,$$

$$\therefore (1\cdot2+5\lambda)^2+(1\cdot6-\cdot6\lambda)^2=4(1+\lambda)^2,$$

which gives $21\cdot36\lambda^2+2\cdot08\lambda=0$.

$$\therefore \lambda(21{\cdot}36\lambda + 2{\cdot}08) = 0,$$

$$\therefore \lambda = 0, \text{ or } -\frac{2{\cdot}08}{21{\cdot}36}.$$

The meaning of these results is easily seen from the diagram below.

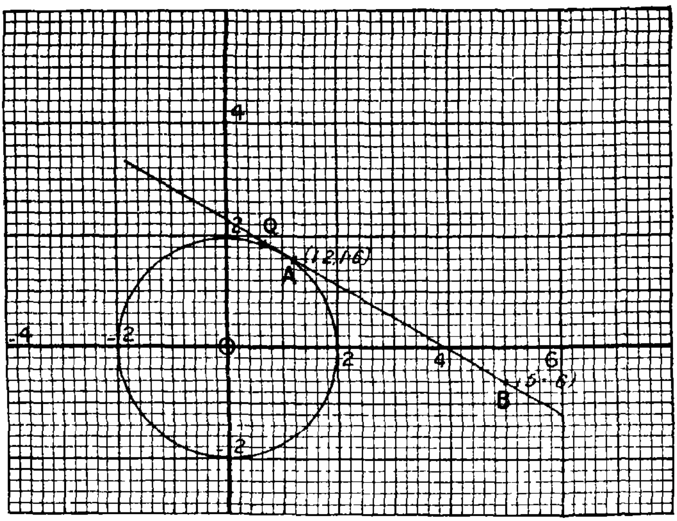


DIAGRAM 1.

We see that the point (1·2, 1·6) itself lies on the circle, so that the line can cut the circle in only one other point.

A segment in one of the two ratios has been reduced to zero, and so we have a zero root in the quadratic equation. To realise how a segment is reduced to zero, look back to the figure of last example and imagine A moved up to P, so that AP becomes of length zero.

(ii.) Let us take the ratio of the segments of the division to be $(1 : \lambda)$ and see what happens.

We have $\qquad x' = \dfrac{1{\cdot}2\lambda + 5}{\lambda+1}$ and $y' = \dfrac{1{\cdot}6\lambda - {\cdot}6}{\lambda+1}$.

Since, as before, $x'^2 + y'^2 = 4$,

$$\therefore \left(\frac{1{\cdot}2\lambda+5}{\lambda+1}\right)^2 + \left(\frac{1{\cdot}6\lambda - {\cdot}6}{\lambda+1}\right)^2 = 4,$$

$$\therefore (1{\cdot}2\lambda+5)^2 + (1{\cdot}6\lambda - {\cdot}6)^2 = 4(\lambda^2+1)^2,$$

$$\therefore 2{\cdot}08\lambda + 21{\cdot}36 = 0.$$

Now we know that we ought to obtain a quadratic equation as a circle cuts a straight line at two points.

It follows that the coefficient of λ^2 must therefore be zero. That is to say, the full equation is

$$0\lambda^2+2\cdot08\lambda+21\cdot36=0.$$

One root of the equation is therefore infinite. (Art. IV.)

This is easily accounted for.

The equation of method (i.) gives the ratio in the form $\frac{0}{PB}(=0)$, causing the absolute term to disappear in the quadratic, while the equation of method (ii.) gives the ratio in the form $\frac{PB}{0}(=\infty)$, causing the coefficient of λ^2 to disappear.

The form of the quadratic equation giving the segments will therefore depend on whether we write $(\lambda : 1)$ or $(1 : \lambda)$.

Example 3.—*In what ratio is the line joining the points $(-3, 4)$ and $(7, -1)$ divided by the circle $x^2+y^2=5$?*

Let the circle divide the join in the ratio $(\lambda : 1)$ at the point (x', y').

$$\therefore x'=\frac{-3+7\lambda}{1+\lambda} \text{ and } y'=\frac{4-\lambda}{1+\lambda}.$$

But $x'^2+y'^2=5$,

$$\therefore (-3+7\lambda)^2+(4-\lambda)^2=5(1+\lambda)^2,$$

whence $9\lambda^2-12\lambda+4=0$,

$$\therefore (3\lambda-2)^2=0.$$

Hence both values of λ are $\frac{2}{3}$.

Since the roots of the equation are coincident it follows that the points of section must be coincident. The line is therefore a tangent to the circle. (See Diagram 2.)

Example 4.—*In what ratio does the circle $x^2+y^2=5$ divide the line joining the points $(-1, 2)$ and $(-5, 0)$?*

(i.) Let the circle cut the join in the ratio $(\lambda : 1)$ at a point (x', y').

Then $$x'=\frac{-1-5\lambda}{1+\lambda} \text{ and } y'=\frac{2}{1+\lambda}.$$

But $x'^2+y'^2=5$,

$$\therefore (-1-5\lambda)^2+(2)^2=5(1+\lambda)^2,$$

which gives $\lambda^2=0$.

Hence both values of λ are zero.

Now the point $(-1, 2)$ lies on the circle, so that the line can cut the

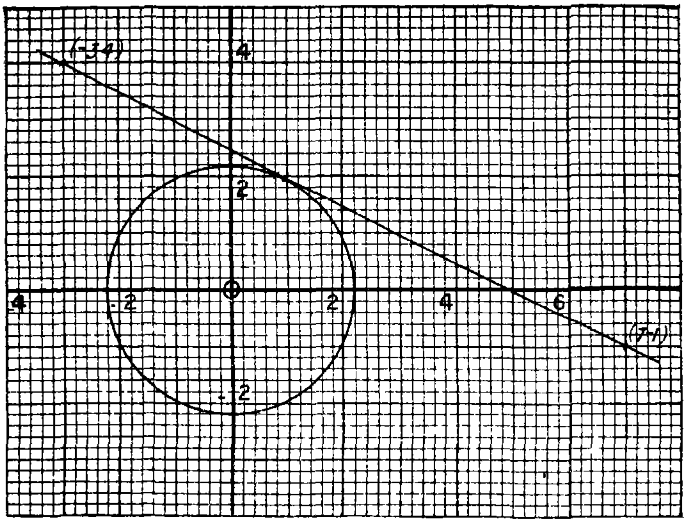


Diagram 2.

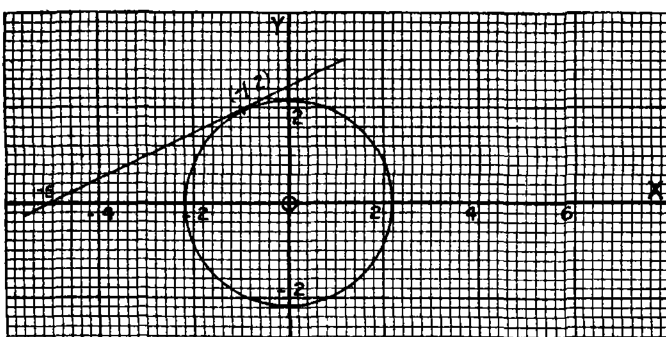


Diagram 3.

circle again in one point only. This point must therefore coincide with $(-1, 2)$, thus reducing both the segmental ratios to zero.

The line therefore touches the circle at the point $(1, 2)$. (Diagram 3.)

(ii.) Let us take the ratio of division as $(1 : \lambda)$.

Then we ought to obtain a quadratic equation of the form

$$0\lambda^2 + 0\lambda + c = 0,$$

that is, $c = 0$. This must happen because the roots of the equation in last method were zero, so that those of the one we shall find must be ∞. The difference is that of obtaining the ratio in the form $\frac{0}{AP}$ and $\frac{AP}{0}$.

We have $\quad x' = \frac{-5-\lambda}{1+\lambda}$ and $y' = \frac{2\lambda}{1+\lambda}$.

Since $x'^2 + y'^2 = 5$,

$$\therefore \left(\frac{-5-\lambda}{1+\lambda}\right)^2 + \left(\frac{2\lambda}{1+\lambda}\right)^2 = 5,$$

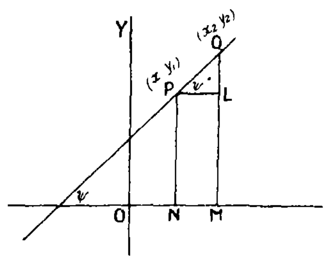
which gives, on simplifying, $20 = 0$.

Hence the coefficients of λ^2 and λ must both be zero in the quadratic. In full the equation is

$$0\lambda^2 + 0\lambda + 20 = 0.$$

Both roots are therefore infinite, so that two of the segments of division—the denominators in the ratios are both zero; and since $(1, 2)$ lies on the circle the other point of section must coincide with it, so that, as we saw before, the circle touches the line at the point $(1, 2)$.

VI. *To express the co-ordinates of a point in terms of those of another point, the slope of the line joining them, and the distance between them.*



Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, and $PQ = r$.

Then $PL = PQ \cos \psi$.

$\therefore x_2 - x_1 = r \cos \psi$,

$\therefore x_2 = x_1 + r \cos \psi. \qquad (1).$

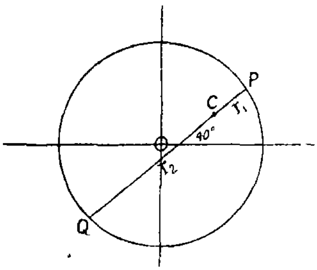
Again, $LQ = PQ \sin \psi$.

$\therefore y_2 - y_1 = r \sin \psi$,

$\therefore y_2 = y_1 + r \sin \psi \qquad (2).$

These formulae are often of very great use in cases where the distance between two points is concerned.

Example 1.—*Through the point $C\equiv(2, 1)$ a straight line is drawn at slope of 40°, so as to cut the circle whose centre is O and radius is 4 units at P and Q. Find the length of CP and of CQ.*



Let the straight line cut the circle at a point (x_1, y_1) at a distance of r units from C.

Then

$$\left.\begin{aligned} x_1 &= 2+r\cos 40^\circ \\ \text{and } y_1 &= 1+r\sin 40^\circ \end{aligned}\right\} \text{(See formulae just found.)}$$

The equation to the circle is

$$x^2+y^2=16.$$

Since (x_1, y_1) lies on it,

$$\therefore\ (2+r\cos 40^\circ)^2+(1+r\sin 40^\circ)^2=16,$$

$$\therefore\ (\cos^2 40^\circ+\sin^2 40^\circ)r^2+2(2\cos 40^\circ+\sin 40^\circ)r-11=0.$$

Now $\cos^2\theta+\sin^2\theta=1$ for all values of θ, as is shown in any book on trigonometry.

$$\therefore\ r^2+2(2\cos 40^\circ+\sin 40^\circ)r-11=0,$$

$$\therefore\ r=1{\cdot}8, \text{ or } -6{\cdot}1 \text{ (approx.)}.$$

Hence $CP=1{\cdot}8$ units and $CQ=6{\cdot}1$ units (nearly).

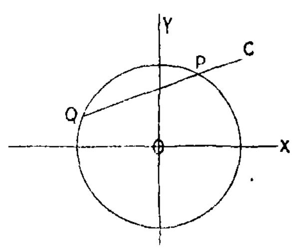
The difference of sign in the values of r is due to the fact that CP and CQ are oppositely directed, that is, have opposite senses.

Example 2.—*Through a fixed point C any straight line is drawn cutting a circle at P and Q. Prove CP·CQ constant.*

Let the centre of the circle be taken as origin, and let its radius be a units in length.

The equation to the circle is therefore

$$x^2+y^2=a^2.$$



Let $C\equiv(h, k)$.

Let the slope of CQ be ψ, and suppose that the line cuts the circle at a point (x_1, y_1) at a distance of r units from C.

Then $\quad x_1=h+r\cos\psi,$

$\qquad\quad y_1=k+r\sin\psi.$

But (x_1, y_1) lies on the circle,

$$\therefore\ x_1^2+y_1^2=a^2,$$

$$\therefore\ (h+r\cos\psi)^2+(k+r\sin\psi)^2=a^2,$$

$$\therefore\ r^2(\cos^2\psi+\sin^2\psi)+2(h\cos\psi+k\sin\psi)r+(h^2+k^2-a^2)=0$$

$$\therefore\ r^2+2(h\cos\psi+k\sin\psi)r+(h^2+k^2-a^2)=0.$$

If r_1 and r_2 be the roots of this equation, then

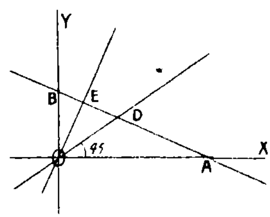
$$r_1 r_2 = h^2 + k^2 - a^2,$$

that is,

$$CP \cdot CQ = h^2 + k^2 - a^2 = \text{constant}.$$

VII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*A straight line is drawn cutting the axes at $A \equiv (a, o)$ and $B \equiv (o, b)$. The bisector of $X\hat{O}Y$ meets AB at D. Prove $AD : DB = OA : OB$.*



Let OD cut AB at D so that

$$AD : DB = \lambda : 1 \quad . \quad (1).$$

Let $D \equiv (x', y')$.

Then, since $A \equiv (a, o)$ and $B \equiv (o, b)$, we have

$$x' = \frac{a}{1+\lambda} \text{ and } y' = \frac{\lambda b}{1+\lambda} \quad \text{(Art. II.)}.$$

Now the equation to OD is $x = y$ (since $X\hat{O}D = 45°$),

$$\therefore x' = y',$$

$$\therefore \frac{a}{1+\lambda} = \frac{\lambda b}{1+\lambda},$$

$$\therefore a = \lambda b,$$

$$\therefore \frac{OA}{OB} = \lambda = \frac{AD}{DB} \text{ by (1)}.$$

EXAMPLE 2.—*In last figure, if OE is perpendicular to AB, prove $AE : EB = OA^2 : OB^2$.*

Let $\frac{AE}{EB} = \lambda$, and let $E \equiv (x', y')$.

The equation to AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Then $\frac{x}{b} - \frac{y}{a} = 0$ is the equation to a line through O perpendicular to AB (Chap. III.).

It is therefore the equation to OE.

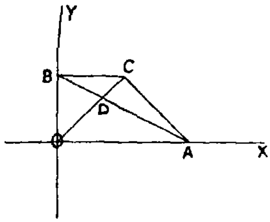
Now $x' = \frac{a}{1+\lambda}$ and $y' = \frac{\lambda b}{1+\lambda}$, and since (x', y') lies on OE,

$$\therefore \frac{a}{b(1+\lambda)} - \frac{\lambda b}{a(1+\lambda)} = 0,$$

$$\therefore a^2 = \lambda b^2,$$

$$\therefore \frac{OA^2}{OB^2} = \lambda = \frac{AE}{EB}.$$

EXAMPLE 3.—*OACB is a trapezium such that $OA \equiv (2c, o)$, $C \equiv (c, c)$, and $B \equiv (o, c)$. Prove that its diagonals cut each other at a point of trisection.*



Let the diagonals intersect at D.

Let $D \equiv (x', y')$, and let $OD = \lambda DC$.

Then, since $O \equiv (o, o)$ and $C \equiv (c, c)$,

$$\therefore x' = \frac{\lambda c}{1+\lambda} \text{ and } y' = \frac{\lambda c}{1+\lambda}.$$

Now the equation to AB is

$$\frac{x}{2c} + \frac{y}{c} = 1$$

$$\text{or } x + 2y = 2c,$$

$$\therefore \frac{\lambda c}{1+\lambda} + \frac{2\lambda c}{1+\lambda} = 2c,$$

which gives $\lambda = 2$.

Hence $OD = 2DC$.

D is therefore a point of trisection on OC.

Similarly, it can be shown to be a point of trisection on AB.

EXAMPLE 4.—*Find the centroid of the triangle whose vertices are the points $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, and $C \equiv (x_3, y_3)$.*

At the centroid of a triangle the medians are divided in the ratio 2 : 1.

Let the centroid of the triangle in question be $G \equiv (x', y')$. Now the mid-point of the join of $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ is

$$F \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Since G is the centroid of the triangle,

$$\therefore CG : GF = 2 : 1,$$

$$\therefore x' = \frac{x_3 + 2 \times \frac{x_1 + x_2}{2}}{1+2} = \frac{x_1 + x_2 + x_3}{3}.$$

Similarly, $$y' = \frac{y_1 + y_2 + y_3}{3}.$$

EXAMPLE 5.—*OACB is a rectangle such that $A \equiv (a, o)$ and $B \equiv (o, b)$. P and Q are taken on AB so that $AP = BQ$. If OP cuts AC at R and OQ cuts BC at S, prove AB parallel to RS.*

The gradient of AB is

$$-\frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

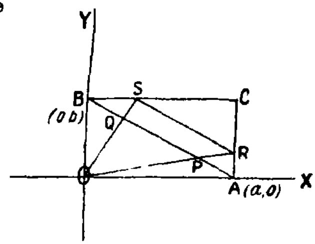
Let $AP : PB = m : n$.

Then $AQ : QB = n : m$,

$$\therefore P \equiv \left(\frac{na}{m+n}, \frac{mb}{m+n}\right) \text{ and } Q \equiv \left(\frac{ma}{m+n}, \frac{nb}{m+n}\right).$$

The equation to OP is therefore

$$y=\frac{mb}{na}x \quad \text{(Chap. III.)}.$$



Now at R, which lies on OP, we have $x=a$,

$$\therefore y=\frac{mb}{n} \quad . \quad . \quad (2).$$

Similarly, the equation to OQ is

$$y=\frac{nb}{ma}x.$$

At S, which lies on OQ, we have $y=b$,

$$\therefore x=\frac{ma}{n} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence, by (2) and (3),

$$R\equiv\left(a, \frac{mb}{n}\right) \text{ and } S\equiv\left(\frac{ma}{n}, b\right).$$

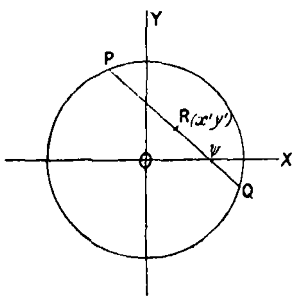
The gradient of RS is therefore

$$\frac{\frac{mb}{n}-b}{a-\frac{ma}{n}} \quad \text{(Chap. III.)}$$

$$=\frac{(m-n)b}{(n-m)a}=-\frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Since the gradients of AB and RS are equal, by results (1) and (4), these lines are therefore parallel.

EXAMPLE 6.—*Find the locus of the mid-points of a system of parallel chords of a circle.*



Let the equation to the circle be

$$x^2+y^2=a^2.$$

Let the slope of the system be ψ.

Let $R\equiv(x', y')$ be the mid-point of one of the chords.

Let the chord cut the circle at a point (x_1, y_1).

$$\therefore x_1=x'+r\cos\psi,$$

$$\text{and } y_1=y'+r\sin\psi.$$

But $x_1^2+y_1^2=a^2$,

$$\therefore (x'+r\cos\psi)^2+(y'+r\sin\psi)^2=a^2,$$

$$\therefore (\cos^2\psi+\sin^2\psi)r^2+2(x'\cos\psi+y'\sin\psi)r+x'^2+y'^2-a^2=0,$$

$$\therefore r^2+2(x'\cos\psi+y'\sin\psi)r+x'^2+y'^2-a^2=0.$$

Now since R is the mid-point of the chord PQ, therefore RP and RQ are equal in length, but are oppositely directed. Hence the roots of the quadratic equation in r must be equal in value but opposite in sign.

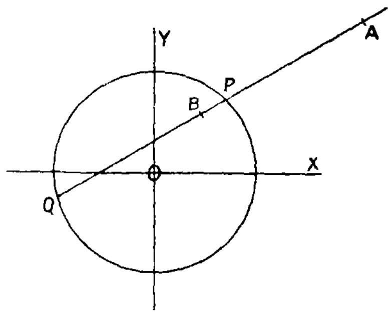
$$\therefore x' \cos\psi + y' \sin\psi = 0 \quad \text{(Art. IV.)}.$$

Hence the locus of (x', y') is the straight line $x \cos\psi + y \sin\psi = 0$. (See Chap. V. Art. VII. Ex. 6, and Chap. VI. Art. I. Ex. 3.)

EXAMPLE 7.—*The line joining the points $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ cuts the circle $x^2 + y^2 = a^2$ at P and Q. Find the condition that P and Q shall harmonically separate A and B.*

Let the circle divide the join of AB at a point (x', y') in the ratio $(\lambda : 1)$.

Then
$$x' = \frac{x_1 + \lambda x_2}{1 + \lambda} \text{ and } y' = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$



But $x'^2 + y'^2 = a^2$,

$$\therefore \left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right)^2 + \left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right)^2 = a^2,$$

$$\therefore (x_1 + \lambda x_2)^2 + (y_1 + \lambda y_2)^2 = a^2(1 + \lambda)^2,$$

$$\therefore (x_2^2 + y_2^2 - a^2)\lambda^2 + 2(x_1 x_2 + y_1 y_2 - a^2)\lambda + x_1^2 + y_1^2 - a^2 = 0.$$

Now P and Q harmonically separate A and B, hence the roots of this equation are equal in value but opposite in sign.

$$\therefore x_1 x_2 + y_1 y_2 - a^2 = 0 \quad \text{(Art. IV.)},$$
$$\text{or } x_1 x_2 + y_1 y_2 = a^2.$$

RÉSUMÉ

1. If $P \equiv (x', y')$ divides the join of $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ in the ratio $m : n$, then

$$x' = \frac{mx_2 + nx_1}{m + n} \text{ and } y' = \frac{my_2 + ny_1}{m + n}.$$

Corollary.—If the ratio is $(\lambda : 1)$, then

$$x' = \frac{x_1 + \lambda x_2}{1 + \lambda} \text{ and } y' = \frac{y_1 + \lambda y_2}{1 + \lambda}$$

2. If P divides AB internally and Q divides it externally in the same ratio, then P and Q "harmonically separate" A and B, and are called "harmonic conjugates" with respect to A and B.

If $\frac{AP}{PB} = \frac{m}{n}$, then $\frac{AQ}{QB} = -\frac{m}{n}$, so that $\frac{AP}{PB} = -\frac{AQ}{QB}$.

Corollary.—A and B harmonically separate P and Q.

3. If $AB = r$, and the slope of AB is ψ,

$$\text{then } x_2 = x_1 + r \cos \psi,$$
$$\text{and } y_2 = y_1 + r \sin \psi.$$

EXAMPLES

1. Plot on squared paper the points (2, 2) and (4, 6).
Find from first principles (*i.e.* as in the theory of Art. II.) the co-ordinates of the point where the join is divided internally in the ratio 2 : 3.

2. The line joining the points (1, 3) and (5, 7) is divided internally at a point P in the ratio 3 : 5. Calculate the co-ordinates of P.

3. The line joining the points (2, 3) and (4, 9) is divided externally at a point P in the ratio 3 : 7.
Find the co-ordinates of P.

4. Find the points which divide the join of $(-1, 2)$ and $(3, 4)$ externally and internally in the ratio (4, 1). Plot the system.

5. In what ratio does the point (2, 6) divide the join of $(-2, 0)$ and (0, 3) ?

6. In what ratio does the line $2x+3y=6$ cut the join of the points $(-1, -3)$ and (2, 4) ? (Art. II. Ex. 4.)

7. In what ratio does the line $3x+5y+10=0$ cut the join of the points (2, 3) and (5, 5) ?

8. Prove that the join of the points $(-2, -4)$ and (3, 3) is cut harmonically by the lines $x+y=2$ and $x+y=18$.

9. Where does the join of the points $(-7, 3)$ and $(2, -3)$ cut the line $3x-4y=35$? Find the point by the method of Art. II. Ex. 4, and by working out the equation to the line through the given points.

10. A straight line cuts the axes at A and B. The bisector of $X\hat{O}Y$ cuts it at C. Prove that $AC:CB=OA:OB$.

11. In last example find D the harmonic conjugate of C with respect to A and B.

12. $ABCD$ is a rectangle. E is the mid-point of BC. Prove that AC cuts DE at a point of trisection.

13. In what ratios does the circle $x^2+y^2=25$ cut the join of the points $(-2, 3)$ and $(1, 6)$?
Draw a squared-paper diagram.

14. In what ratios does the circle $x^2+y^2=6$ cut the join of the points $(2, 1)$ and $(5, 3)$?

15. In what ratios does the circle $x^2+y^2=13$ cut the join of the points $(2, 3)$ and $(1, -4)$?
Verify by drawing the graphs.

16. In what ratios does the circle $x^2+y^2=10$ cut the join of the points $(1, 3)$ and $(-2, 4)$?

What do you conclude? (See Art. V. Ex. 4.)
Draw a squared-paper diagram.

17. Show by intersection ratios that the circle $x^2+y^2=8$ touches the join of the points $(7, -3)$ and $(0, 4)$ at the point $(2, 2)$.
Draw a squared-paper diagram.

18. Show that the circle $x^2+y^2=5$ touches the join of the points $(1, -3)$ and $(5, 5)$.
Draw a representation on squared paper.

19. Prove that the circle $x^2+y^2=20$ divides the join of the points $(4, 6)$ and $(-1, 4)$ harmonically. Draw a diagram and verify by measurement.

20. Prove that the circle $x^2+y^2=9$ divides the join of the points $(-1, 2)$ and $(3, 6)$ harmonically.
Verify by a diagram.

21. Through the point $P\equiv(5, 2)$ a straight line of slope 32° is drawn. Q is a point 2 units further up the line. Work out the co-ordinates of Q by the method of Article VI.

22. The slope of a line is 142° and $P\equiv(3, 4)$ is a point on it. Q is taken on the line at a distance of 3 units from P. Find the co-ordinates of Q.

23. Through the point $(4, 2)$ a system of straight lines is drawn whose members cut the circle $x^2+y^2=25$.
Prove that the product of the segments is constant. (Art. VII. Ex. 2.)

24. Through the point $(-5, 3)$ any straight line is drawn cutting the circle $x^2+y^2=16$. Find the product of the segments.

25. ABC is a triangle such that $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, and $C \equiv (x_3, y_3)$. Assuming that the centroid of a triangle is such as to divide a median in the ratio 2 : 1, find the centroid of $\Delta\, ABC$.

26. Find the centroid of the triangle the equations to whose sides are $y = o$, $2x = y + 4$ and $3x + y = 21$.

27. Find the centroid of the triangle made by the axes and the line $\frac{x}{a} + \frac{y}{b} = 1$.

28. Find the locus of the mid-points of the system of chords of the circle $x^2 + y^2 = 9$, which are parallel to the line $2x = 5y$.

29. Find the locus of the mid-points of a system of chords of the circle $x^2 + y^2 = 36$, whose gradients are $\cdot 8$.

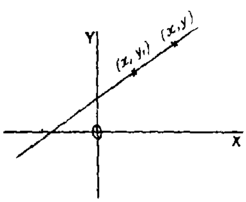
30. Through the point $C = (2, 4)$ a variable straight line is drawn cutting the axes at A and B. Prove that the locus of P the harmonic conjugate of C with respect to A and B is a straight line.

(*Hints.*—Take the equation to AB in intercept form. Let C divide AB as $\lambda : 1$. Then P will divide it as $-\lambda : 1$. Express the co-ordinates of C and P by the ratio formulae, and eliminate a, b, and λ.)

CHAPTER VIII

OTHER FORMS OF THE EQUATION TO A STRAIGHT LINE: CONCURRENCY AND COLLINEARITY

I. *Find the equation to the straight line of gradient m which passes through the point (x_1, y_1).*



Let (x, y) be any point on the straight line.

Then the gradient of the line through (x_1, y_1) and (x, y) is $\frac{y - y_1}{x - x_1}$ (Chap. III.),

$$\therefore \frac{y - y_1}{x - x_1} = m,$$

$$\therefore y - y_1 = m\ (x - x_1).$$

EXAMPLE 1.—*Write down the equation to the straight line which passes through the point* (2, 5) *and has a gradient of* $\frac{2}{3}$.

We have $x_1 = 2$, $y_1 = 5$, and $m = \frac{2}{3}$,

$$\therefore y - 5 = \tfrac{2}{3}(x - 2).$$

Whence $2x - 3y + 11 = 0$.

EXAMPLE 2.—*Find the equation to the straight line through the point* $(2, -3)$ *which is parallel to the line* $2x - 4y = 3$.

The gradient of $2x - 4y = 3$ is $\frac{1}{2}$.

The gradient of the required line is therefore $\frac{1}{2}$ also. (Chap. III.)

Hence we have $x_1 = 2$, $y_1 = -3$, and $m = \frac{1}{2}$.

$$\therefore y + 3 = \tfrac{1}{2}(x - 2),$$

$$\therefore x - 2y = 8.$$

EXAMPLE 3.—*Find the equation to the straight line passing through the point* $(-4, 3)$ *and perpendicular to* $x - 2y = 8$.

Since the gradient of $x - 2y = 8$ is $\frac{1}{2}$ therefore that of the required line is $-\frac{2}{1}$ (Chap. III.).

The required equation is therefore

$$y-3=-2(x+4),$$
$$\therefore\ 2x+y+5=0.$$

II. *Equation to a pencil of lines passing through a given point* (x_1, y_1).

Consider the equation

$$y-y_1=m(x-x_1).$$

It is evident that every time we assign a new value to m we obtain a different line. Nevertheless each one of these lines will pass through the point (x_1, y_1).

The system of lines we obtain by giving different values to m is called the "pencil" of lines having (x_1, y_1) as vertex, and m is called the "parameter" of the system.

The parameter is fixed for any one line, but varies from line to line.

EXAMPLE 1. —*Draw members of the pencil of lines whose vertex is at the point* (2, 5).

We have $y-5=m(x-2)$ as the equation to some line passing through the point (2, 5).

(1) Let $m=1 \therefore y-5=1(x-2) \therefore x-y+3=0.$
(2) Let $m=\frac{1}{2} \therefore y-5=\frac{1}{2}(x-2) \therefore 3x-2y+4=0.$
(3) Let $m=\frac{4}{5} \therefore y-5=\frac{4}{5}(x-2) \therefore 4x-5y+17=0.$
(4) Let $m=-\frac{2}{3} \therefore y-5=-\frac{2}{3}(x-2) \therefore 2x+3y-19=0.$
(5) Let $m=0 \therefore y-5=0(x-2) \therefore y-5=0.$

In this way we can obtain an infinity of lines.

The case when the slope is 90°, so that the gradient m (or tan 90°) is infinite, must be noticed.

We have $\frac{1}{m}(y-5)=x-2.$

Now when m is ∞, then $\frac{1}{m}=0.$

$$\therefore\ x-2=0.$$

The line $x-2=0$ is one of the system. (See Diagram 1.)

EXAMPLE 2.—*Interpret the equation* $y=mx+b$ *in the light of this article.*

We can write $y-b=m(x-0)$.

When m varies the equation is that of a pencil of lines whose vertex is (o, b).

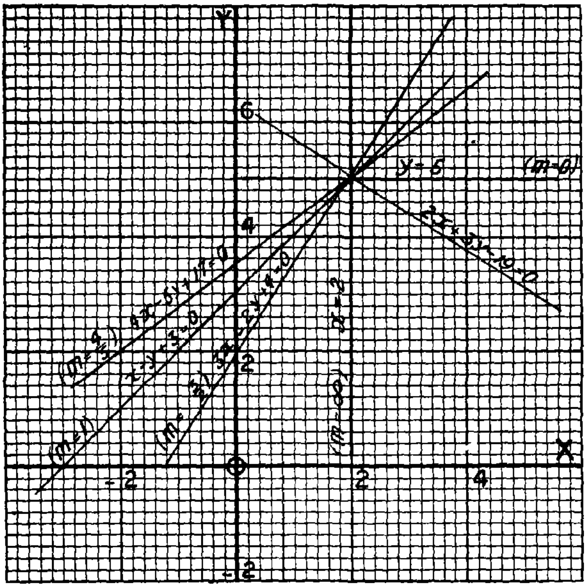


DIAGRAM I.

III. *Find the equation to the straight line which passes through the points* (x_1, y_1) *and* (x_2, y_2).

Let (x, y) be any other point on the line.

Now the gradient of the line is $\frac{y_2 - y_1}{x_2 - x_1}$ (Chap. II.).

But we may also derive the gradient by considering the line as the join of (x, y) and (x_1, y_1).

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

each ratio measuring the gradient of the line.

Last equation may be written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

The latter form is that in which the equation is usually quoted.

EXAMPLE 1.—*Find the equation to the straight line which passes through the points* (1, 2) *and* (3, 5).

We have $x_1=1$, $x_2=3$, $y_1=2$, and $y_2=5$,

$$\therefore \frac{y-2}{5-2}=\frac{x-1}{3-1},$$

$$\therefore \frac{y-2}{3}=\frac{x-1}{2},$$

$$\therefore 3x-2y+1=0.$$

EXAMPLE 2.—*Find the equation to the straight line joining the origin to the point* (5, 8).

We have $x_1=0$, $y_1=0$, $x_2=5$, and $y_2=8$,

$$\therefore \frac{x-0}{5-0}=\frac{y-0}{8-0},$$

$$\text{or } \frac{x}{5}=\frac{y}{8}.$$

Compare this result with the method of Chapter II.

IV. *Collinearity of points.*

EXAMPLE 1.—*Find the condition that the three points* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *may be collinear.*

The equation to the line through (x_1, y_1) and (x_2, y_2) is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

Now if (x_3, y_3) is a point on this line, then

$$\frac{y_3-y_1}{y_2-y_1}=\frac{x_3-x_1}{x_2-x_1}.$$

This is the condition sought.

Thus we find the equation to the line through two of the points, and see if the co-ordinates of the third point satisfy it.

EXAMPLE 2.—*Show that the points* $(-4, 1)$, $(2, 4)$, *and* $(6, 6)$ *are collinear.*

The line joining the first two points is

$$\frac{y-1}{4-1}=\frac{x+4}{2+4}.$$

If (6, 6) lies on this line then

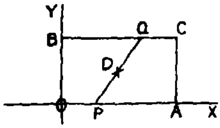
$$\frac{6-1}{4-1}=\frac{6+4}{2+4},$$

$$\text{that is } \frac{5}{3}=\frac{10}{6},$$

which is true, hence the three points are collinear.

Example 3.—*C is a point in the plane of the axes. CA and CB are drawn perpendicular to XX' and YY' respectively. P is taken on OA and Q on BC so that $OP = CQ$. Prove that PQ passes through the intersection of the diagonals of the rectangle $OACB$.*

Let $A \equiv (a, o)$, $B \equiv (o, b)$ and therefore $C \equiv (a, b)$.



Let $OP = c$, $\therefore BQ = a - c$.

$\therefore Q \equiv (a - c, b)$.

The diagonals intersect at D the mid-point of OC,

$$\therefore D \equiv \left(\frac{a}{2}, \frac{b}{2}\right).$$

The equation to the line joining $P \equiv (c, o)$ and $Q \equiv (a - c, b)$ is

$$\frac{y}{b} = \frac{x - c}{a - c - c} \quad \text{(Art. III.)}.$$

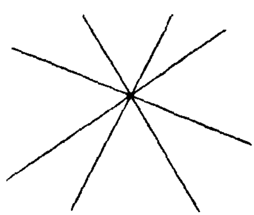
If $D \equiv \left(\frac{a}{2}, \frac{b}{2}\right)$ is a point on this line then

$$\frac{b/2}{b} = \frac{a/2 - c}{a - 2c},$$

$$\text{that is } \frac{1}{2} = \frac{a - 2c}{2(a - 2c)}$$
$$= \tfrac{1}{2}$$

which is true, showing that the three points are collinear.

V. *Concurrent Lines.*—A number of lines are concurrent when they all pass through the same point.



Hence if we wish to test whether three lines $l_1x + m_1y + n_1 = 0$,

$$l_2x + m_2y + n_2 = 0,$$

and $l_3x + m_3y + n_3 = 0$ are concurrent, we have merely to find the point of intersection of any pair and see whether its co-ordinates satisfy the equation to the third.

Example.—*Show that the lines $x - 2y = 1$, $3x - 2y - 11$, and $4x + 3y = 26$ are concurrent.*

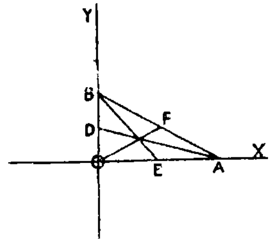
On solving the first two equations we obtain

$$x = 5 \text{ and } y = 2.$$

If the point (5, 2) lies on the third line $4x + 3y = 26$, then must $20 + 6 = 26$.

This being true, it follows that all three lines pass through the point (5, 2).

EXAMPLE 2.—*A straight line cuts the axes at A and B. Prove that the medians of triangle OAB are concurrent.*



Let $A \equiv (2a, o)$, $B \equiv (o, 2b)$, so that $E \equiv (a, o)$, $D \equiv (o, b)$ and $F \equiv (a, b)$.

The equation to AD is

$$\frac{x}{2a} + \frac{y}{b} = 1 \text{ (Chap. II.).}$$

The equation to BE is

$$\frac{x}{a} + \frac{y}{2b} = 1.$$

The equation to OF is

$$\frac{x}{a} = \frac{y}{b} \text{ (present Chap.).}$$

Solve the second and third equations.

We have, on eliminating x, $\frac{y}{b} + \frac{y}{2b} = 1$.

$$\therefore y = \frac{2b}{3},$$

$$\therefore x = \frac{2a}{3}.$$

The point $\left(\frac{2a}{3}, \frac{2b}{3}\right)$ lies on the first line $\frac{x}{2a} + \frac{y}{b} = 1$,

$$\text{if } \frac{1}{3} + \frac{2}{3} = 1.$$

Hence all three lines pass through the point $\left(\frac{2a}{3}, \frac{2b}{3}\right)$

VI. *Equation to a system of lines passing through the intersection of two given lines.*

Take the lines $x - 2y - 1 = 0$ and $3x - 2y - 11 = 0$.

On solving the equations we find that their point of intersection is (5, 2).

Consider now the equation

$$(x - 2y - 1) + \lambda(3x - 2y - 11) = 0 \quad . \quad . \quad (1).$$

On collecting terms we have

$$(1 + 3\lambda)x - 2(1 + \lambda)y - (1 + 11\lambda) = 0 \quad . \quad (2).$$

Form (2) shows us that the equation is that of a straight line.

Form (1) shows us that the equation is satisfied by those values of x and y, which make $x-2y-1=0$ and $3x-2y-11=0$ simultaneously. That is to say, it is satisfied by $x=5$ and $y=2$. The point (5, 2) therefore lies on the graph of equation (1).

Hence we conclude that the graph of

$$(x-2y-1)+\lambda(3x-2y-11)=0$$

is a straight line passing through (5, 2), the point of intersection of the lines $x-2y-1=0$ and $3x-2y-11=0$.

By giving various values to λ we obtain a system of lines all passing through the point (5, 2). (Diagram 2.)

For example put $\lambda=-2,\ -1,\ +1,\ +3$ in succession.

We have

(1) $x-2y-1-2(3x-2y-11)=0$ that is $5x-2y-21=0$.

(2) $x-2y-1-1(3x-2y-11)=0$ that is $x-5=0$.

(3) $x-2y-1+1(3x-2y-11)=0$ that is $x-y-3=0$.

(4) $x-2y-1+3(3x-2y-11)=0$ that is $5x-4y-6=0$.

If we put $\lambda=0$ we have the given line $x-2y-1=0$.

Again, if we write the equation as follows

$$\frac{1}{\lambda}(x-2y-1)+(3x-2y-11)=0,$$

and put $\lambda=\infty$, then since $\frac{1}{\infty}=0$ we obtain the equation $3x-2y-11=0$, which is that of the second given line.

The system of lines found by assigning different values to λ is called the "pencil" of lines having (5, 2) as "vertex."

λ is called the "parameter" of the system. It is fixed for any one line, but varies from line to line.

The lines $x-2y-1=0$ and $3x-2y-11=0$ are called the "base lines" of the system and their parameters are $\lambda=0$ and $\lambda=\infty$ respectively.

The method of reasoning is quite general. Let

$$l_1x+m_1y+n_1=0 \text{ and } l_2x+m_2y+n_2=0$$

be the equations of two given lines.

Then $l_1x+m_1y+n_1+\lambda(l_2x+m_2y+n_2)=0$ is the equation

to a straight line passing through the intersection of the two given lines. For if we write the equation thus:

$$(l_1 + \lambda l_2)x + (m_1 + \lambda m_2)y + (n_1 + \lambda n_2) = 0,$$

we see that the equation is represented by a straight line, while the first form shows us that it passes through the point of intersection of the given pair, as the co-ordinates of the latter

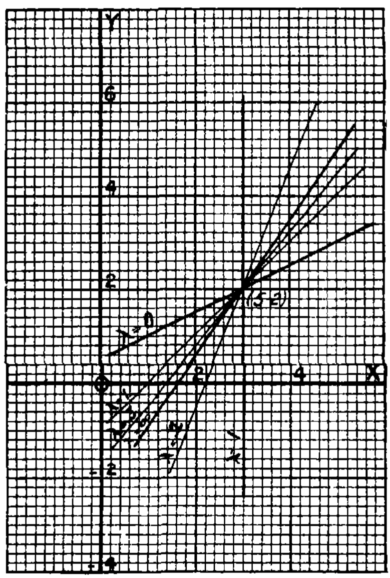


DIAGRAM 2.

point cause the expressions $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ to vanish. As shown above we can obtain a whole set of lines all passing through the intersection of the given pair simply by varying λ. As already stated the system of lines found is called a pencil of lines, whose base lines are $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ and having λ as parameter. The parameter of the first base line is $\lambda = 0$ and of the second $\lambda = \infty$.

It falls to be noted that the equation

$$y - y_1 = m(x - x_1)$$
$$\text{or } (y - y_1) - m(x - x_1) = 0,$$

is a simple case of what we have been dealing with.

Here the equation represents a pencil of lines passing through the intersection of the lines $y - y_1 = 0$ and $x - x_1 = 0$.

The point where these two base lines intersect is of course (x_1, y_1), the vertex of the pencil.

EXAMPLE 1.—*Find the equation to the straight line which joins the origin to the intersection of the lines $2x - 5y + 4 = 0$ and $3x - y + 1 = 0$.*

The line $2x - 5y + 4 + \lambda\ (3x - y + 1) = 0$ passes through the intersection of the given pair.

It also passes through the origin if $4 + \lambda(1) = 0$,

$$\therefore \lambda = -4.$$

The required equation is therefore

$$2x - 5y + 4 - 4(3x - y + 1) = 0$$
$$\text{or } 10x + y = 0.$$

NOTE.—The two given lines intersect at the point $(-\frac{1}{13}, \frac{10}{13})$ which plainly lies on $10x + y = 0$.

EXAMPLE 2.—*Find the equation to the straight line which passes through the intersection of the lines $x - 2y = 3$ and $3x - y = 4$ and has a gradient of $\frac{3}{4}$.*

$x - 2y - 3 + \lambda(3x - y - 4) = 0$ is the equation to a straight line passing through the intersection of the given pair. On collecting terms we have

$$(1 + 3\lambda)x - (2 + \lambda)y - (3 + 4\lambda) = 0.$$

The gradient of this line is $\dfrac{1 + 3\lambda}{2 + \lambda}$,

$$\therefore \frac{1 + 3\lambda}{2 + \lambda} = \frac{3}{4}.$$

Whence $\lambda = \frac{2}{9}$,

$$\therefore x - 2y - 3 + \tfrac{2}{9}\ (3x - y - 4) = 0,$$

which gives $3x - 4y - 7 = 0$.

EXAMPLE 3.—*Find the equation to the straight line which is perpendicular to $5x + 2y - 3 = 0$ and passes through the intersection of the lines $2x - 4y + 3 = 0$ and $x - y = 2$.*

Since the required line is perpendicular to $5x + 2y - 3 = 0$ its gradient is $\frac{2}{5}$ (Chap. III.).

Let its equation be

$$2x - 4y + 3 + \lambda\ (x - y - 2) = 0,$$
$$\therefore (2 + \lambda)x - (4 + \lambda)y + (3 - 2\lambda) = 0.$$

The gradient of this line is $\frac{2+\lambda}{4+\lambda}$,

$$\therefore \frac{2+\lambda}{4+\lambda}=\frac{2}{5},$$

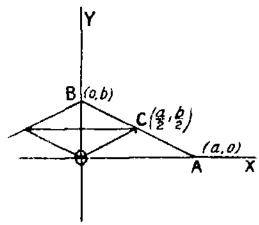
$$\therefore \lambda=-\tfrac{2}{3}.$$

The equation sought is by the usual process

$$4x-10y+13=0.$$

EXAMPLE 4.—*A straight line is drawn cutting the axes at A and B. C is the mid-point of AB. Through C a parallel is drawn to XX′, through B a parallel is drawn to OC, and through O a parallel is drawn to AB. Prove that these three lines are concurrent.*

Let $A\equiv(a,\ o)$, $B\equiv(o,\ b)$, and therefore $C\equiv\left(\frac{a}{2},\ \frac{b}{2}\right)$.



The equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

$\therefore \frac{x}{a}+\frac{y}{b}=0$ is the equation to a line through O parallel to AB (Chap. III.).

The gradient of OC is $\frac{b}{a}$.

$\therefore y=\frac{b}{a}x+b$ is the equation to a straight line through B parallel to OC (Chap. III.).

On simplifying these two equations we have

$$bx+ay=0 \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and } bx-ay+ab=0 \quad . \quad . \quad . \quad (2).$$

$\therefore (bx+ay)-(bx-ay+ab)=0$ (Note. $\lambda=-1$) is the equation to a line through their point of intersection.

This gives $2ay-ab=0$,

$$\text{or } y=\frac{b}{2}.$$

But this last equation is that of the straight line through C parallel to XX'.

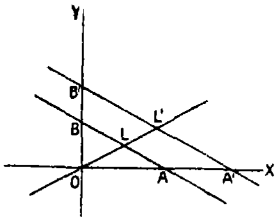
Hence the three lines are concurrent.

VII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Two parallel straight lines cut the axes at A, B, A′ and B′ respectively. If L and L′ are the mid-points of AB and A′B′ respectively prove O, L and L′ collinear.*

Let the equation to AB be

$$\frac{x}{a}+\frac{y}{b}=1.$$



That of $A'B'$ will therefore be

$$\frac{x}{a}+\frac{y}{b}=k \text{ (Chap. III.).}$$

Hence $A\equiv(a, o)$, $B\equiv(o, b)$, $A'=(ka, o)$ and $B'\equiv(o, kb)$,

$$\therefore L\equiv\left(\frac{a}{2}, \frac{b}{2}\right) \text{ and } L'\equiv\left(\frac{ka}{2}, \frac{kb}{2}\right).$$

The equation to LL' is

$$\frac{y-\frac{b}{2}}{\frac{kb}{2}-\frac{b}{2}}=\frac{x-\frac{a}{2}}{\frac{ka}{2}-\frac{a}{2}}$$

$$\therefore \frac{2y-b}{b(k-1)}=\frac{2x-a}{a(k-1)},$$

$$\therefore \frac{2y-b}{b}=\frac{2x-a}{a},$$

which gives

$$\frac{y}{b}=\frac{x}{a}.$$

This equation shows that LL' passes through O.

Example 2.—*Tangents are drawn to the circle $x^2+y^2=a^2$ at the points (x_1, y_1) and (x_2, y_2) lying on it. Prove that the line joining O to their point of intersection is perpendicular to the chord of contact.*

The tangent at (x_1, y_1) is $xx_1+yy_1=a^2$.

The tangent at (x_2, y_2) is $xx_2+yy_2=a^2$.

$\therefore (xx_1+yy_1-a^2)+\lambda\ (xx_2+yy_2-a^2)=0$ is a line passing through their point of intersection. (Art. VI.)

It passes through the origin if $-a^2-\lambda a^2=0$.

That is, if $\lambda=-1$,

$$\therefore (xx_1+yy_1-a^2)-(xx_2+yy_2-a^2)=0,$$

$$\therefore (x_1-x_2)x+(y_1-y_2)y=0.$$

The gradient of the line is $-\frac{x_1-x_2}{y_1-y_2}$.

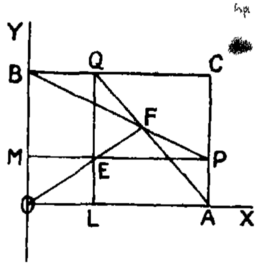
The gradient of the chord of contact which joins (x_1, y_1) and (x_2, y_2) is $\frac{y_1-y_2}{x_1-x_2}$ (Chap. III.).

The product of the gradients is -1.

Therefore the two lines are perpendicular.

Example 3.—*In the accompanying figure OACB is a rectangle. LEQ and MEP are perpendiculars to the axes. Prove BP, AQ, and OE concurrent.*

Let $L\equiv(l, o)$ and $A\equiv(a, o)$.



Let $M \equiv (o, m)$ and $B \equiv (o, b)$,

$\therefore$ $P \equiv (a, m)$, $Q \equiv (l, b)$, and $E \equiv (l, m)$.

The equation to AQ is

$$\frac{y}{b} = \frac{x-a}{l-a} \text{ (Art. III.).}$$

The equation to BP is

$$\frac{y-b}{m-b} = \frac{x}{a}.$$

On arranging these equations we have

$$bx - (l-a)y - ab = 0$$
$$\text{and } (m-b)x - ay + ab = 0.$$

$\therefore$ $\{(bx - (l-a)y - ab\} + \{(m-b)x - ay + ab\} = 0$ is the equation to a straight line passing through the point of intersection of AQ and BP.

Simplification of this last equation gives

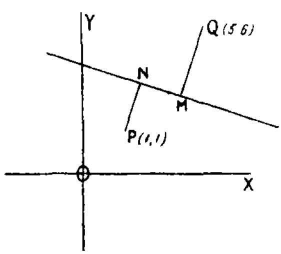
$$mx - ly = 0$$

$$\text{or } \frac{x}{l} = \frac{y}{m}.$$

This is the equation to OE.

Hence AQ, BP and OE are concurrent.

EXAMPLE 4.—*A variable straight line is drawn so as to pass between the points $P \equiv (1, 1)$ and $Q \equiv (5, 6)$. If the ratio of the perpendiculars from P and Q to the line be always $1 : 2$, prove that it passes through a fixed point.*



Let the equation to the variable line be

$$y = mx + c \quad . \quad (1).$$

$$\therefore \; PN = \frac{m - 1 + c}{\sqrt{m^2+1}} \text{ (numerically : Chap. IV.).}$$

$$\text{and } QM \equiv \frac{5m - 6 + c}{\sqrt{m^2+1}}.$$

Now these two expressions must have contrary signs since P and Q lie on opposite sides of the line (Chap. IV.).

But $\frac{PN}{PM} = \frac{1}{2}$,

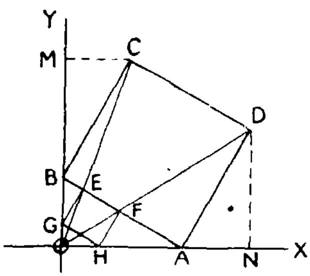
$$\therefore \; \frac{1}{2} = -\frac{m-1+c}{5m-6+c}.$$

Whence $c = 4 - \frac{7}{2}m$,

$$\therefore \; y = mx + 4 - \tfrac{7}{2}m \text{ (by substitution in (1))}$$
$$\text{or } y - 4 = m(x - \tfrac{7}{2}).$$

Hence the variable line always passes through the fixed point $(\frac{7}{2}, 4)$ (Art. II.).

EXAMPLE 5.—*A straight line cuts the axes at $A \equiv (4, 0)$ and $B \equiv (0, 3)$. On AB a square $ABCD$ is drawn so as to be turned away from O. OC and OD cut AB at E and F respectively. The perpendicular from E to AB cuts YY' at G, and that from F cuts XX' at H. Prove GH parallel to AB.*



It can easily be proved that triangles AND, OAB, and BMC are congruent.

$$\therefore ND = BM = OA = 4$$
$$\text{and } AN = MC = OB = 3,$$
$$\therefore ON = 7 \text{ and } OM = 7,$$
$$\therefore D \equiv (7, 4) \text{ and } C \equiv (3, 7).$$

Hence the equation to OC is

$$\frac{x}{3} = \frac{y}{7} \text{ or } 7x - 3y = 0.$$

But the equation to AB is

$$\frac{x}{4} + \frac{y}{3} = 1 \text{ or } 3x + 4y - 12.$$

Therefore $3x + 4y - 12 + \lambda(7x - 3y) = 0$ is a line through the intersection of AB and OC.

It is perpendicular to AB if its gradient is $\frac{4}{3}$ (Chap. III.).

Arranging last equation we have

$$(3 + 7\lambda)x + (4 - 3\lambda)y - 12 = 0.$$

Hence if this is the equation to EG we must have

$$-\frac{3 + 7\lambda}{4 - 3\lambda} = \frac{4}{3} \text{ (by gradients),}$$

$$\therefore \lambda = -\tfrac{25}{9},$$

$$\therefore \text{the equation to } EG \text{ is } 148x - 111y + 108 = 0,$$

$$\therefore G \equiv (0, \tfrac{108}{111}).$$

Similarly $H \equiv (\frac{192}{148}, 0)$,

$$\therefore \text{the gradient of } GH \text{ is } -\frac{108/111}{192/148} = -\frac{3}{4},$$

$$\therefore GH \text{ is parallel to } AB.$$

EXAMPLE 6.—*Through the fixed point $A \equiv (h, k)$ a straight line is drawn cutting XX' at P. Through the same point another line is drawn perpendicular to the first, cutting the y-axis at Q. Find the locus of the mid-point of PQ.*

Let the equation to AP be

$$y - k = m(x - h) \quad . \quad . \quad . \quad . \quad (1).$$

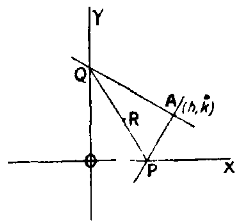
Then the gradient of AQ will be $-\dfrac{1}{m}$ and its equation will be

$$y - k = -\frac{1}{m}(x - h) \quad . \quad . \quad . \quad (2).$$

To find P we have from equation (1) that when $y=0$ then $-k=m(x-h)$,

$$\therefore\ x=\frac{mh-k}{m},$$

$$\therefore\ P\equiv\left(\frac{mh-k}{m},\ 0\right) \quad . \quad (3).$$



To find Q we have from equation (2) that when $x=0$ then

$$y-k=\frac{h}{m},$$

$$\therefore\ y=\frac{h+mk}{m},$$

$$\therefore\ Q\equiv\left(0,\ \frac{h+mk}{m}\right) \quad . \quad . \quad . \quad (4).$$

Let $R\equiv(x',\ y')$ be the mid-point of PQ,

$$\therefore\ \text{by (3) and (4) } x'=\frac{mh-k}{2m} \text{ and } y'=\frac{h+mk}{2m}.$$

Now m is a variable depending on the slope of the variable line AP, therefore it must not appear in the equation to the locus.

We have $2mx'=mh-k$, giving

$$m=-\frac{k}{2x'-h}$$

and $2my'=h+mk$, giving

$$m=\frac{h}{(2y'-k)},$$

$$\therefore\ \frac{-k}{2x'-h}=\frac{h}{2y'-k},$$

$$\text{or } -k(2y'-k)=h(2x'-h),$$

that is

$$\left(y'-\frac{k}{2}\right)=-\frac{h}{k}\left(x'-\frac{h}{2}\right).$$

The locus of $R\equiv(x',\ y')$ is the straight line $y-\frac{k}{2}=-\frac{h}{k}\left(x-\frac{h}{2}\right)$.

It passes through $\left(\frac{h}{2},\ \frac{k}{2}\right)$, the mid-point of OA, and is perpendicular to OA, whose gradient is $\frac{k}{h}$ (Chap. III.).

N.B.—*R is the centre of a variable circle passing through O and A.*

RÉSUMÉ

1. $y-y_1=m(x-x_1)$ is the equation to a straight line of gradient m passing through the point $(x_1,\ y_1)$.

2. If m varies a "pencil of lines" having (x_1, y_1) as vertex is obtained.

3. $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}$ is the equation to a straight line passing through the points (x_1, y_1) and (x_2, y_2).

4. $l_1x+m_1y+n_1+\lambda(l_2x+m_2y+n_2)=0$ is the equation to a straight line passing through the intersection of the lines

$$l_1x+m_1y+n_1=0,$$
$$l_2x+m_2y+n_2=0.$$

5. If λ is made to vary a pencil of lines results. λ is called the "parameter" of the system.

EXAMPLES

1. Write down the equations to the straight lines which pass through the following points and have the gradients given.

(i.) Point (4, 1) : gradient $\frac{3}{2}$.
(ii.) Point $(-3, 2)$: gradient $-\frac{4}{3}$.
(iii.) $(-3{\cdot}3, -2{\cdot}8)$: gradient $\frac{4}{3}$.
(iv.) $(2{\cdot}5, -1{\cdot}4)$: gradient $-$[illegible].

Draw the first two lines.

2. Work out the equation to a straight line which passes through the point (3, 1) and makes a negative intercept of 4 units on the y-axis.

3. Through the point $D\equiv(h, k)$ a straight line is drawn perpendicular to OD.
Write down its equation and find the points P and Q where it crosses the axes.

4. P is a variable point on the x-axis and Q one on the y-axis. They are joined to the point $H\equiv(5, 3)$, and are such that $P\widehat{H}Q$ is right.
Prove that the locus of the mid-point of PQ is a straight line.

5. Work out the equations to the lines which join the following pairs of points.

(i.) (1, 2) and (4, 5).
(ii.) $(-2, 3)$ and (4, 6).
(iii.) $(-5, -2)$ and $(1, -1)$.
(iv.) (h, k) and (o, b).

6. Draw members of the pencil of lines whose vertex is at the point $(-3, 1)$ and show the values of the parameter.
What are the base lines of the system and what values has the parameter in their case?

7. Write down the equation to a system of lines passing through the intersection of the lines $x-2y+1=0$ and $3x-8y+9=0$.

Find the parameters of the following members:

(i.) That which passes through the origin.
(ii.) That which has a gradient of $-\frac{1}{3}$.
(iii.) That which is parallel to the line $5x+2y=12$.
(iv.) That which is perpendicular to the member through the origin.

8. Write down the equation to a system of lines passing through the intersection of the lines

$$2x-5y=3,$$
$$3x+4y=16.$$

(i.) Which member of the system passes through the origin?
(ii.) Which member is perpendicular to last line?
(iii.) Which member makes a negative intercept of unit length on the x-axis?
(iv.) Which member has a gradient of $\frac{1}{2}$?
(v.) Which member is parallel to the y-axis?
(vi.) Which members of the system make with the axes triangles whose areas are 12·5 sq. units?
(vii.) Through what point do all the members pass? Verify the result for the straight lines found above.

Draw several members of the above system of lines, showing the values of the parameter.

9. Write down the equation to a system of lines passing through the intersection of the lines $x+2y=9$ and $3x-5y=5$.

(i.) Which member makes a positive intercept of 4 units on the y-axis?
(ii.) Which member is perpendicular to last line?
(iii.) Find the area of the quadrilateral in the first quadrant made by these lines and the axes.

Draw several members of the above pencil showing the values of the parameter.

10. Prove that the medians of a right-angled triangle are concurrent.

11. Show that the following systems of points are collinear.

(i.) $(-1,\ -2)$, $(3,\ 2)$ and $(4,\ 3)$,
(ii.) $(-3,\ -1)$, $(0,\ 4)$ and $(-6,-6)$,
(iii.) $(-5,\ 3)$, $(2,\ 0)$ and $(9-3)$.

12. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{a}+\frac{y}{b}=\lambda_1$, and $\frac{x}{a}+\frac{y}{b}=\lambda_2$ are three straight lines which cut the axes at A, B, A_1, B_1, and A_2, B_2 respectively.

Prove that the mid-points of AB, A_1B_1 and A_2B_2 are collinear.

13. ***ABCD* is a square. *BA* is produced to *E*, so that $AE = AB$. *CE* is joined and cuts *AD* at *G*. The diagonal *AC* is divided at *F* in the ratio 1 : 2.**

Prove that G, F, and B are collinear.

14. A point moves so that the ratio of the perpendiculars drawn from it to the two lines $2x - 5y = 13$ and $3x + 4y = 16$ is always 5. Prove that its locus is a straight line passing through the intersection of the given pair. (Art. VI.)

15. Two parallel straight lines cut the axes at A, B and A', B' respectively. M is the mid-point of AB. Prove AB', $A'B$ and OM concurrent.

16. The straight line $2x + y = 4$ cuts the axes at A and B. A triangle PQR is drawn such that the equation to PQ is $y = -3$, the equation to QR is $2x + y = 12$ and the equation to RP is $x = -1$.

Prove that PO, QA, and RB are concurrent.

17. A line is drawn through the point $\left(-c, \dfrac{2c}{t}\right)$ perpendicular to the line $t^2x - ty + c = 0$. Prove that it passes through a fixed point for all values of t.

(*Hint.*—Use Art. I., obtaining the gradient from the given line.)

18. Find the condition that the following three lines be concurrent.

$$x + 2y = 9,$$
$$3x - 5y = 5,$$
$$\alpha x + \beta y = 1.$$

19. OAB is a triangle such that $O \equiv (o, o)$, $A \equiv (o, a)$, and $B \equiv (h, k)$. Find the equation to each of its altitudes and prove these three lines concurrent.

20. In last example find also the equations to the lines joining each vertex to the mid-point of the opposite side and so prove the medians concurrent.

21. $ABCD$ is a square. Any line is drawn through A, and perpendiculars DQ and BP are drawn to it. Parallels are drawn to the axes through P and Q so that a rectangle is formed. Prove that AC passes through one of its vertices and BD through the other.

(*Hint.*—Take the equation to the line through A in gradient form.)

22. Two adjacent sides of a rectangle are produced their own lengths. Prove that the extremities of these lines are collinear with a vertex of the rectangle.

23. $ABCD$ is a rectangle such that $AB = h$ and $AD = k$. P, R, Q, and S are taken on AB, BC, CD, and DA respectively so that $AP = p$, $BR = r$, $DQ = q$, and $AS = s$.

(i.) Take A as origin, AB and AD as axes of x and y respectively and work out the equations to PQ and RS.

(ii.) Prove that if these lines intersect on BD then $BP \cdot SD = DQ \cdot BR$.

(*Hint.*—Use Art. VI. and work out the two conditions that B and D lie on a line through their intersection. Between these eliminate the parameter.)

(iii.) Show that if they intersect on AC then $AP \cdot RC = AS \cdot QC$.

24. $ABCD$ is a rectangle. Points K and L are taken on AB and AD respectively and through them parallels LEN and KEM are drawn to the sides. Prove that BM, DN and AE are concurrent.

25. In the figure of last example prove also that LM, KN, and AC are concurrent.

26. Prove for the figure of last example that LK, MN. and DB are concurrent.

27. Any point R is taken on the bisector of $X\hat{O}Y$.
A is a given point on OX and B on OY.
AR meets OY in Q and BR meets OX in P.
Prove that PQ passes through a fixed point on the bisector of $X'\hat{O}Y$.

28. ABC is a right-angled triangle whose hypotenuse is BC. Squares are described externally on AB and AC. B is joined to the corner of the square on AC opposite to it and C to that of the square on AB opposite to it. Prove that these two lines intersect on the perpendicular to BC.

29. Prove that the mid-points of the non-parallel sides of a trapezium and the intersection of its diagonals are collinear.

30. Prove that the non-parallel sides, and the straight line joining the mid-points of the parallel sides of a trapezium are concurrent.

31. $ABCD$ is a rectangle. F is taken on AB and H on AD, and perpendiculars FEG and HEK are drawn to the opposite sides. Prove that the following systems of lines are concurrent.

(1) DE, AK, and CF.
(2) HC, BE, and AG.
(3) BH, DF, and CE.
(4) DK, BG, and AE.

32. Any point P is taken on the bisector of $X\hat{O}Y$. PN is drawn perpendicular to OX. N is joined to the point $H \equiv (1, 3)$ and NH cuts OY at L. Prove that LP always passes through the point $(1, 4)$.

CHAPTER IX

THE HOMOGENEOUS EQUATION OF THE SECOND DEGREE IN x AND y

I. *The degree of an equation.*

The degree of the product $x^p y^q z^r$ is $p+q+r$.

For example the degree of x^2yz^4 is the seventh.

The presence of a numerical factor in the product does not affect its degree.

Thus the degree of $5x^p y^q z^r$ is still $p+q+r$.

The degree of an equation is that of its highest term as defined above.

For example the degree of the equation

$$x^3+2xy+3y^2=0$$

is the third, since x^3 is the term of highest degree.

Again the equation $x^4y+2x^2y^2-4y=0$ is of the fifth degree, the term of highest degree being x^4y.

II. *Homogeneous equations.*

When all the terms of an equation are of the same degree it is said to be homogeneous.

Thus the equation $x^3+4x^2y-3xy^2+5y^3=0$ is homogeneous since all its terms are of the third degree.

Again the equation $5x^3-2x^2z+4yz^2-6xyz=0$ is a homogeneous equation of the third degree in x, y, and z.

Such an equation as $3x^2-4xy+5y^2=0$ is a homogeneous equation of the second degree in x and y.

The equation $ax^2+2hxy+by^2=0$ is called the general homogeneous equation of the second degree in x and y.

In it a, h, and b are assigned numbers.

III. *To discuss the equation* $3x^2+5xy+y^2=0$ *and its grap*

We shall draw its graph from $x=-3$ to $x=+3$. (Diagram 1.)

We note with our first glance that the origin is a point on the graph.

When $x=-3$ we have $27-15y+y^2=0$, which gives

$$y=12{\cdot}91 \text{ or } 2{\cdot}09.$$

When $x=-2$ then $12-10y+y^2=0$, which gives

$$y=8{\cdot}61 \text{ or } 1{\cdot}39.$$

We notice that two values of y correspond to a given value of x.

Proceeding as above we derive the following table :

x	-3	-2	-1	0	1	2	3
y_1	$12{\cdot}91$	$8{\cdot}61$	$4{\cdot}30$	0	$-4{\cdot}30$	$-8{\cdot}61$	$-12{\cdot}91$
y_2	$2{\cdot}09$	$1{\cdot}39$	$\cdot 70$	0	$-\cdot 70$	$-1{\cdot}39$	$-2{\cdot}09$

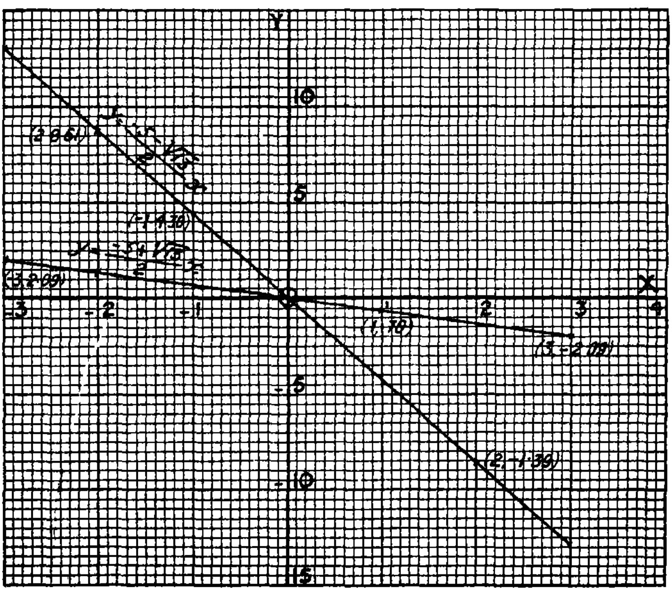


DIAGRAM 1.

We see that the graph consists of two straight lines.

It can be proved to do so very easily.

Regard the given equation as a quadratic in y and solve in the usual way.

We have
$$y^2+5xy+3x^2=0,$$
$$\therefore y=\frac{-5x\pm\sqrt{13x^2}}{2}$$
$$=\frac{-5x\pm x\sqrt{13}}{2}.$$

The solutions therefore are
$$y=\frac{-5+\sqrt{13}}{2}x \quad . \quad . \quad . \quad (1)$$
$$\text{and } y=\frac{-5-\sqrt{13}}{2}x \quad . \quad . \quad . \quad (2).$$

Each of these equations is represented graphically by a straight line passing through the origin (Chap. II). Their gradients are very nearly $-\cdot 697$ and $-4\cdot 303$.

On drawing their graphs they will be found to be the pair of lines already obtained.

To impress these ideas we shall work out another example, say the graph of $2x^2-2xy-y^2=0$.

We have
$$y^2+2xy-2x^2=0.$$

Hence when $x=-3$, then
$$y^2-6y-18=0,$$
which gives $y=8\cdot 20$ or $-2\cdot 20$ (nearly).

Thus we see as before that two values of y correspond to an assigned value of x.

Proceeding as above we have the following table.

x	-3	-2	-1	0	1	2	3
y_1	$8\cdot 20$	$5\cdot 46$	$2\cdot 73$	0	$-2\cdot 73$	$-5\cdot 46$	$-8\cdot 20$
y_2	$-2\cdot 20$	$-1\cdot 46$	$-0\cdot 73$	0	$0\cdot 73$	$1\cdot 46$	$2\cdot 20$

On drawing the graph we obtain as before a pair of straight lines. (Diagram 2.)

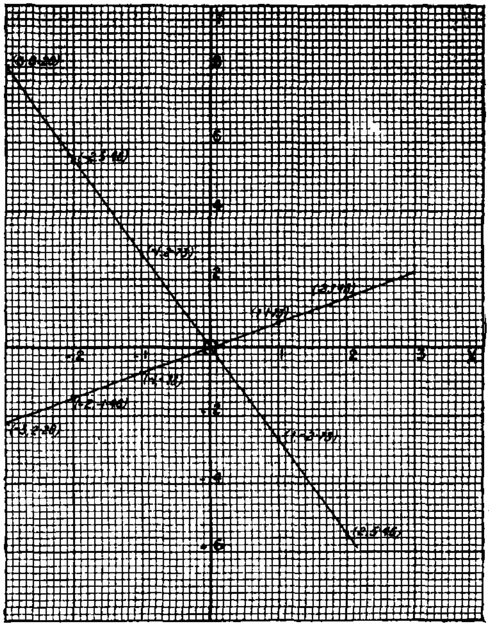


DIAGRAM 2

We can prove that this must be so just as before

Since $y^2 + 2xy - 2x^2 = 0$,

$$\therefore y = \frac{-2x \pm \sqrt{12x^2}}{2} = (-1 \pm \sqrt{3})x$$

The solutions are $y = (-1 + \sqrt{3})x$ and $y = (-1 - \sqrt{3})x$.

The graph of each of these equations is a straight line through the origin

Their graph tables are appended and show that the straight lines are those already obtained.

$y = (-1 + \sqrt{3})x$

x	-3	-2	-1	0
y	$-2\ 20$	$-1\ 46$	$-0\ 73$	0

$y = (-1 - \sqrt{3})x$

x	-3	-2	-1	0
y	$8\ 20$	$5{\cdot}46$	$2{\cdot}73$	0

IV. *A homogeneous equation of the second degree in x and y is graphically represented by a pair of straight lines passing through the origin.*

Let $ax^2+2hxy+by^2=0$ be the homogeneous equation of the second degree in x and y.

Regard it as a quadratic in y,

$$\therefore by^2+2hxy+ax^2=0,$$

$$\therefore y=\frac{-2hx\pm\sqrt{h^2x^2-4abx^2}}{2b}$$

$$=\frac{-hx\pm x\sqrt{h^2-ab}}{b}.$$

The solutions therefore are

$$y=\frac{-h+\sqrt{h^2-ab}}{b}x \quad . \quad . \quad . \quad (1)$$

$$\text{and } y=\frac{-h-\sqrt{h^2-ab}}{b}x \quad . \quad . \quad . \quad (2).$$

The graph of each of these equations is a straight line through the origin.

Hence the homogeneous equation $ax^2+2hxy+by^2=0$ is graphically represented by a pair of lines passing through the origin.

The gradients of the two branches of the graph are respectively

$$\frac{-h+\sqrt{h^2-ab}}{b} \text{ and } \frac{-h-\sqrt{h^2-ab}}{b} \text{ (Chap. III.)}$$

Corollary.—The lines are perpendicular if

$$\frac{-h+\sqrt{h^2-ab}}{b}\times\frac{-h-\sqrt{h^2-ab}}{b}=-1 \text{ (Chap. III.)},$$

that is, if $$\frac{h^2-(h^2-ab)}{b^2}=-1,$$

that is, if $$\frac{ab}{b^2}=-1,$$

which gives $$a+b=0.$$

Hence the lines representing $ax^2+2hxy+by^2=0$ are at right

angles if the algebraic sum of the coefficients of x^2 and y^2 is zero.

EXAMPLE 1.—*Of what two straight lines is* $3x^2-5xy-2y^2=0$ *the joint equation?*

Solving for x we have

$$x=\frac{5y\pm\sqrt{25y^2+24y^2}}{6}=\frac{5y\pm 7y}{6}.$$

The solutions are $x=2y$

and $x=-\frac{1}{3}y.$

Hence the two branches of the graph are the lines

$$x-2y=0$$

and $3x+y=0.$

EXAMPLE 2.—*What two straight lines compose the graph of*

$$8x^2-22xy+15y^2=0\,?$$

Here as wherever possible we solve the equation by the method of factors.

Factorising the left hand member of the equation we have

$$(2x-3y)(4x-5y)=0,$$

$$\therefore\ 2x-3y=0 \text{ or } 4x-5y=0.$$

These two equations are those of the required straight lines.

EXAMPLE 3.—*Prove from elementary principles and by the help of the corollary of this article that the two straight lines which represent the equation* $10x^2-21xy-10y^2=0$ *are at right angles to one another.*

(i.) On factorising we have

$$(2x-5y)(5x+2y)=0.$$

The solutions are

$$2x-5y=0$$

and $5x+2y=0.$

In these equations the coefficients of x and y are interchanged and the sign of one of them reversed, which proves the lines at right angles.

(ii.) Using the corollary of this article we have the sum of the coefficients of x^2 and y^2 $=10-10=0.$

Hence the lines are perpendicular.

EXAMPLE 4.—*Find the combined equation to the straight lines which pass through the origin and the points* (2, 1) *and* (3, 4).

(i.) By Chapters II. and VIII. the equations to the lines joining the origin and the points (2, 1) and (3, 4) are

$$x-2y=0 \text{ and } 4x-3y=0.$$

Hence their combined equation is

$$(x-2y)(4x-3y)=0,$$

$$\therefore\ 4x^2-11xy+6y^2=0.$$

(ii.) *Aliter.*—Let the required equation be $ax^2+2hxy+by^2=0$. Then since (2, 1) and (3, 4) are points on the graph we have

$$4a+4h+b=0 \quad . \quad . \quad . \quad (1),$$

$$\text{and } 9a+24h+16b=0 \quad . \quad . \quad . \quad (2).$$

Treating these as simultaneous equations in a and h we have

$$a=\tfrac{2}{3}b \text{ and } h=-\tfrac{11}{12}b.$$

Hence on substituting in $ax^2+2hxy+by^2=0$ we have

$$\tfrac{2}{3}bx^2-\tfrac{11}{6}bxy+by^2=0,$$

whence $4x^2-11xy+6y^2=0$ as before.

EXAMPLE 5.—*Find the joint equation to two perpendicular straight lines which pass through the origin, if one of them passes through the point* (1, 4).

(i.) The equation to the straight line passing through the origin and the point (1, 4) is

$$4x-y=0.$$

Hence the line through the origin perpendicular to it is

$$x+4y=0.$$

The joint equation is therefore

$$4x^2+15xy-4y^2=0.$$

(ii.) *Aliter.*—Let their joint equation be

$$ax^2+2hxy+by^2=0.$$

Since they are perpendicular we have

$$a+b=0 \text{ or } b=-a \text{ (Cor. to present Art.)},$$

$$\therefore ax^2+2hxy-ay^2=0.$$

Now (1, 4) is a point on the graph,

$$\therefore a+8h-16a=0,$$

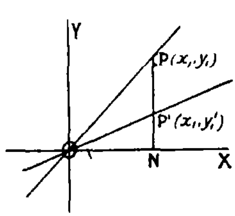
$$\therefore h=\tfrac{15}{8}a,$$

$$\therefore ax^2+\tfrac{15}{4}axy-ay^2=0,$$

whence $4x^2+15xy-4y^2=0$.

V. *An equation giving the gradients of the two lines which constitute the graph of* $ax^2+2hxy+by^2=0$.

Let P be a point on one branch of the graph.



Draw $PP'N$ perpendicular to OX (see figure).

Let $P\equiv(x_1, y_1)$ and $P'\equiv(x_1, y'_1)$.

Then the gradients of the lines are $\dfrac{NP}{ON}$ and $\dfrac{NP'}{ON}$, or what is the same thing $\dfrac{y_1}{x_1}$ and $\dfrac{y_1'}{x_1}$.

Now we know that when we assign to x a value x_1 in the equation $ax^2+2hxy+by^2=0$, then two values y_1 and y_1' of y correspond. Hence the gradients are easily found.

Further, if we divide both sides of the equation by x^2 we have

$$b\left(\frac{y}{x}\right)^2+2h\frac{y}{x}+a=0.$$

This equation will therefore give the two ratios $\frac{y_1}{x_1}$ and $\frac{y_1'}{x_1}$, that is to say, it gives the gradients of the lines.

If as usual we write m for $\frac{y}{x}$, the gradient, then

$$bm^2+2hm+a=0$$

is the equation of the gradients.

Corollary.—The lines are perpendicular if $a+b=0$.

Let the roots of last equation be m_1 and m_2. These are therefore the gradients of the two lines.

The lines are perpendicular if $m_1m_2=-1$, that is, if

$$\frac{a}{b}=-1 \text{ (Chap. V. Art VI property 2)}$$

$$\text{or } a+b=0.$$

This result was obtained in last article.

EXAMPLE 1.—*Find the gradients of the lines $2x^2-8xy+3y^2=0$.*

On dividing throughout by x^2 we have

$$3\left(\frac{y}{x}\right)^2-8\frac{y}{x}+2=0,$$

$$\therefore 3m^2-8m+2=0.$$

The solutions—found by the usual process—are $m=2{\cdot}3874$ or 2792 (very nearly).

These are the gradients of the lines.

Corollary.—The slopes of the lines are 67° 16′ and 15° 36′.

EXAMPLE 2.—*Prove from the equation of gradients that the lines $3x^2-4xy-3y^2=0$ are perpendicular.*

We have $3y^2+4xy-3x^2=0$,

$$\therefore 3\left(\frac{y}{x}\right)^2+4\frac{y}{x}-3=0$$

$$\text{or } 3m^2+4m-3=0.$$

Let m_1 and m_2 be the gradients of the two constituent lines.

Then m_1 and m_2 are the roots of the last equation

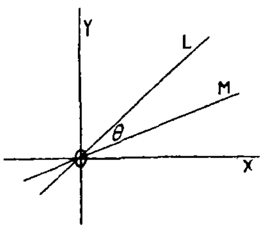
$$\therefore m_1 m_2 = -\tfrac{3}{3}$$
$$= -1.$$

The lines are therefore perpendicular.

VI. *To find the angle between the straight lines*

$$ax^2 + 2hxy + by^2 = 0$$

Let OL and OM be the two constituent lines of the graph.



Let θ be the angle between them, and let their gradients be m_1 and m_2 respectively

Then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \text{ (Chap III).}$$

Now the gradients of the lines are given by the equation

$$bm^2 + 2hm + a = 0 \text{ (see last article).}$$

Therefore m_1 and m_2 are the roots of this equation.

We have

$$(m_1 - m_2)^2 = m_1^2 + m_2^2 - 2m_1 m_2$$
$$= (m_1 + m_2)^2 - 4m_1 m_2.$$

But

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b} \text{ (Chap V Art VI.),}$$

$$\therefore (m_1 - m_2)^2 = \frac{4h^2}{b^2} - \frac{4a}{b}$$
$$= \frac{4(h^2 - ab)}{b^2},$$

$$\therefore m_1 - m_2 = \pm\frac{2\sqrt{h^2 - ab}}{b}.$$

Again $$1 + m_1 m_2 = 1 + \frac{a}{b} = \frac{a+b}{b},$$

$$\therefore \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}.$$

Note.—In practice there will seldom be any need to pay attention to the sign of the formula. The arithmetical value

will give the acute angle between the lines. A negative value would merely give the obtuse angle.

Corollary 1.—The lines are coincident if $h^2 = ab$.

The lines coincide when $\theta = 0$ and therefore $\tan\theta = 0$.

$$\therefore \frac{2\sqrt{h^2 - ab}}{a+b} = 0,$$

$$\therefore h^2 - ab = 0 \text{ or } h^2 = ab$$

Corollary 2.—The lines are perpendicular if $a + b = 0$.

If $\theta = 90°$ then $\tan\theta = \infty$.

$$\therefore \frac{2\sqrt{h^2 - ab}}{a+b} = \infty,$$

$$\therefore a + b = 0.$$

EXAMPLE 1.—*Find the acute angle between the lines*

$$2x^2 - 6xy + 3y^2 = 0,$$

from the equation of gradients and by formula.

(i.) The equation of gradients is $3m^2 - 6m + 2 = 0$

For θ we have $\qquad \tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

Now $\qquad (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$.

But $\qquad m_1 + m_2 = 2$ and $m_1 m_2 = \dfrac{2}{3}$,

$$\therefore (m_1 - m_2)^2 = 4 - \frac{8}{3} = \frac{4}{3},$$

$$\therefore m_1 - m_2 = \frac{2}{\sqrt{3}} \text{ (numerically).}$$

Also $\qquad 1 + m_1 m_2 = 1 + \dfrac{2}{3} = \dfrac{5}{3}$,

$$\therefore \tan\theta = \frac{2\sqrt{3}}{5} = \cdot 6928 \text{ (approx.)},$$

$$\therefore \theta = 34° \ 43'.$$

(ii.) Use the formula :—In this example $a = 2$, $h = -3$, and $b = 3$.

$$\therefore \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{3}}{5},$$

$$\therefore \theta = 34° \ 43' \text{ as before.}$$

EXAMPLE 2.—*Prove by Corollary* (1) *and otherwise that the lines* $4x^2+20xy+25y^2=0$ *are coincident.*

(i.) We saw that the lines $ax^2+2hxy+by^2=0$ are coincident if $h^2=ab$.

The lines $4x^2+20xy+25y^2=0$ will therefore be coincident if $100=100$. Hence the lines coincide.

(ii.) If we factorise the left-hand member we have

$$(2x+5y)(2x+5y)=0,$$

$$\text{or } (2x+5y)^2=0.$$

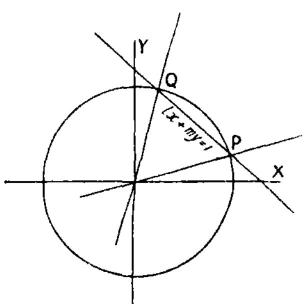
The lines therefore coincide with the line $2x+5y=0$.

NOTE.—*The lines composing the graph of the equation*

$$ax^2+2hxy+by^2=0$$

will coincide when the factors of the expression $ax^2+2hxy+by^2$ *are equal. In other words,* $ax^2+2hxy+by^2$ *must be a perfect square. This is illustrated in the example.*

VII. *Find the combined equation to the straight lines joining the origin to the intersections of the straight line* $lx+my=1$ *and the circle* $x^2+y^2=a^2$.



The analytical device now to be introduced is very important.

Consider the equation

$$x^2+y^2=a^2(lx+my)^2.$$

It is formed by multiplying the right-hand member of the equation to the circle, by the square of the left-hand member of the equation to the straight line which has the value 1 for all points on PQ (see figure).

It is evident that the co-ordinates of the point P will satisfy the equation

$$x^2+y^2=a^2(lx+my)^2. \quad \text{For let } P\equiv(x',\ y').$$

Then since P lies on the circle

$$\therefore\ x'^2+y'^2=a^2,$$

and since P is also on the straight line

$$\therefore\ lx'+my'=1.$$

Hence

$$\begin{aligned} &a^2(lx'+my')^2\\ &=a^2\times 1\\ &=x'^2+y'^2, \end{aligned}$$

which shows that P lies on the graph of $x^2+y^2=a^2(lx+my)^2$.

Similarly it may be proved that Q also is a point on this graph.

If we simplify the equation we have

$$x^2+y^2=a^2(l^2x^2+2lmxy+m^2y^2),$$

which gives $(a^2l^2-1)x^2+2a^2lmxy+(a^2m^2-1)y^2=0$.

Now this last equation is a homogeneous equation of the second degree in x and y, so that its graph is a pair of straight lines passing through the origin.

We saw that P and Q lay on the graph, therefore

$$x^2+y^2=a^2(lx+my)^2$$

is the joint equation to OP and OQ.

Remark.—The equation to the straight line must be in the form $lx+my=1$, that is to say the right-hand member must be $+1$.

EXAMPLE 1.—*Find the equation to the pair of lines which join the origin to the intersections of the straight line $2x-5y-3$ and the circle $x^2+y^2=4$.*

Since $2x-5y=3$,

$$\therefore \tfrac{2}{3}x-\tfrac{5}{3}y=1.$$

The equation to the pair of lines in question is therefore

$$x^2+y^2=4(\tfrac{2}{3}x-\tfrac{5}{3}y)^2.$$

Which gives when simplified

$$7x^2-80xy+91y^2=0.$$

EXAMPLE 2.—*Find the combined equation to the pair of lines which join the origin to the intersections of the straight line $x+2y=20$ and the circle $x^2+y^2=85$. Find where the straight line cuts the circle and so verify that the equation obtained is the right one.*

(i.) We have $x+2y-20$,

$$\therefore \frac{x+2y}{20}=1,$$

$$\therefore x^2+y^2=85\left(\frac{x+2y}{20}\right)^2 \quad . \quad . \quad . \quad (1).$$

This is the joint equation to the two straight lines in question

When simplified it becomes $63x^2-68xy+12y^2=0$.

(ii.) Verification:—On solving the equations

$$x^2+y^2=85$$

$$\text{and } x+2y=20$$

we have the following solutions

x	2	6
y	9	7

The straight line cuts the circle at the points (2, 9) and (6, 7).

The point (2, 9) lies on the graph

$$63x^2 - 68xy + 12y^2 = 0$$

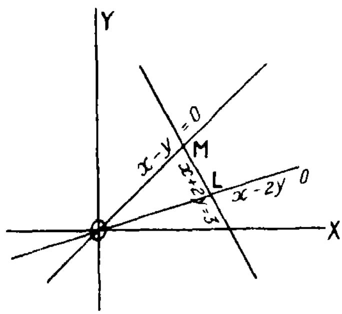
if

$$252 - 1224 + 972 = 0.$$

Similarly it can be shown that the point (6, 7) lies on the graph. The origin obviously is on it.

VIII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Find the orthocentre of the triangle the equations to whose sides are $x^2 - 3xy + 2y^2 = 0$ and $x + 2y = 3$.*



We have $x^2 - 3xy + 2y^2 = 0$,

$$\therefore (x - 2y)(x - y) = 0.$$

Hence the equations to OL and OM are

$$x - 2y = 0 \text{ and } x - y = 0.$$

Now $x - y + \lambda(x + 2y - 3) = 0$ is the equation to a line which passes through the intersection of OM and ML.

Collect terms,

$$\therefore (1 + \lambda)x - (1 - 2\lambda)y - 3\lambda = 0.$$

This line is perpendicular to

$$x - 2y = 0 \text{ or } OL \text{ if}$$

$$1(1 + \lambda) - 2(1 - 2\lambda) = 0 \text{ (Chap. III. Art. I.)}.$$

Whence $\lambda = 1$.

The equation to the perpendicular from M to OL is therefore

$$x - y + 1(x + 2y - 3) = 0$$

$$\text{or } 2x + y - 3 = 0 \quad . \quad . \quad . \quad . \quad (1).$$

Again, since the equation to ML is $x + 2y = 3$, therefore $2x - y = c$ is the equation to a line perpendicular to it (Chap. III.).

When $c = 0$ the line passes through the origin.

Hence $2x - y = 0$ (2) is the equation to the perpendicular to ML from O.

On solving equations (1) and (2) we find that $(\frac{3}{4}, \frac{3}{2})$ is the orthocentre of triangle OLM.

EXAMPLE 2.—*Find the centroid of the triangle the equations to whose sides are $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.*

Since $12x^2 - 20xy + 7y^2 = 0$,

$$\therefore (2x - y)(6x - 7y) = 0.$$

The equations to the sides of the triangle are therefore

$$2x - y = 0 \quad . \quad . \quad . \quad . \quad (1),$$

$$6x - 7y = 0 \quad . \quad . \quad . \quad . \quad (2),$$

$$\text{and } 2x - 3y + 4 = 0 \quad . \quad . \quad . \quad . \quad (3).$$

On solving these equations two and two we find that the vertices of the triangle are (0, 0), (1, 2), and (7, 6).

The mid-point of (0, 0) and (1, 2) is ($\frac{1}{2}$, 1) (Chap. I.). Hence the equation to the median through (7, 6) is

$$\frac{x-7}{\frac{1}{2}-7}=\frac{y-6}{1-6} \quad \text{(Chap. VIII.)},$$

$$\text{whence } 10x-13y=-8 \quad . \quad . \quad . \quad . \quad (4).$$

Similarly, the equation to the median through O is

$$x-y=0 \quad . \quad . \quad . \quad . \quad . \quad (5).$$

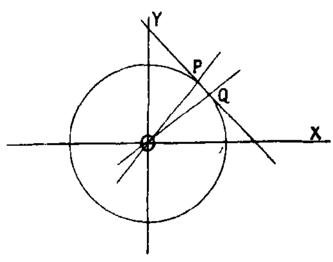
On solving equations (4) and (5) we have that the centroid of the triangle is $(\frac{8}{3}, \frac{8}{3})$.

EXAMPLE 3.—*Find the condition that the line $lx+my=1$ may touch the circle $x^2+y^2=a^2$.*

The equation $x^2+y^2=a^2(lx+my)^2$ is the equation to a pair of lines passing through the origin and the intersections of the line $lx+my=1$ with the circle $x^2+y^2=a^2$ (Art. VII.).

Arranging the equation we have

$$(a^2l^2-1)x^2+2a^2lmxy+(a^2m^2-1)y^2=0 \quad . \quad (1).$$



Now if $lx+my=1$ touches the circle, then its points of intersection with the latter are coincident. Hence the lines joining them to the origin are coincident.

The lines represented by (1) must therefore coincide.

$\therefore\ 4a^4l^2m^2=4(a^2l^2-1)(a^2m^2-1)$,

which gives

$$a^2l^2+a^2m^2=1$$

$$\text{or } l^2+m^2=\frac{1}{a^2}.$$

COR.—*The lines are at right angles if* $l^2+m^2=\frac{2}{a^2}$.

The condition for perpendicularity is that the sum of the coefficients of x^2 and y^2 be zero.

$$\therefore\ a^2l^2-1+a^2m^2-1=0,$$

which gives $$l^2+m^2=\frac{2}{a^2}.$$

Note.—It follows that if OP and OQ are at right angles, then PQ touches the circle $x^2+y^2=\frac{a^2}{2}$.

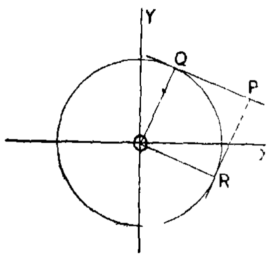
Example 4.—*Find the locus of the intersection of tangents to a circle which are at right angles to one another.*

Let PQ and PR be two perpendicular tangents to the circle.

Then OQ and OR must also be at right angles.

Let the equation to the circle be

$$x^2+y^2=a^2 \text{ and let } P\equiv(x_1,\ y_1).$$



The equation to QR is

$$xx_1+yy_1=a^2 \text{ (Chap. VI.)},$$

$$\therefore \frac{xx_1+yy_1}{a^2}=1.$$

The combined equation to OQ and OR is therefore

$$x^2+y^2=a^2\left(\frac{xx_1+yy_1}{a^2}\right)^2$$

$$=\frac{x^2x_1{}^2+y^2y_1{}^2+2xyx_1y_1}{a^2},$$

that is, $\qquad a^2x^2+a^2y^2=x_2{}^2x_1{}^2+y^2y_1{}^2+2xyx_1y_1,$

whence $\qquad (x_1{}^2-a^2)x^2+2x_1y_1xy+(y_1{}^2-a^2)y^2=0.$

Since these lines are perpendicular the sum of the coefficients of x^2 and y^2 is zero (Art. IV. Corollary).

$$\therefore\ x_1{}^2+y_1{}^2-2a^2=0.$$

The locus of $P\equiv(x_1,\ y_1)$ is the circle whose equation is

$$x^2+y^2=2a^2.$$

(See page 97, Ex. 8, and page 111, Ex. 1.)

RÉSUMÉ

1. The degree of the product $x^py^qz^r$ is $p+q+r$.

2. The degree of an equation is that of its term of highest degree.

3. If the terms are all of the same degree the equation is homogeneous.

4. $ax^2+2hxy+by^2=0$ is the general homogeneous equation of the second degree in x and y, and is graphically represented by a pair of straight lines passing through the origin.

5. The equation $b\left(\frac{y}{x}\right)^2+2h\frac{y}{x}+a=0$ or $bm^2+2hm+a=0$ gives the gradients of the two branches of the graph.

6. The angle between the two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}.$$

7. The lines are coincident if $h^2=ab$ and at right angles if $a+b=0$.

EXAMPLES

1. Draw the graphs of the equations $3x^2+2xy-y^2=0$ and $4x^2-7xy+2y^2=0$.

2. Find the area of the triangle formed by the graphs of $3x^2+4xy-4y^2=0$ and $5x-6y+16=0$.

3. Find the individual equations of the straight lines of which the following are the combined equations:

(i.) $5x^2+13xy-6y^2=0$.
(ii.) $15x^2+22xy+8y^2=0$.
(iii.) $6x^2-5xy=0$.

4. Show that the lines $9x^2+6xy+y^2=0$ are coincident.

5. Show that the pairs of lines $x^2-y^2=0$ and $6x^2+5xy-6y^2=0$ include right angles (Art. IV. Corollary).

6. Find h in order that the lines $4x^2+hxy+25y^2=0$ may be coincident.

7. Find b in order that the lines $8x^2+3hxy+by^2=0$ may be perpendicular.

8. Write out the equations which give the gradients of the following lines (Art. V.).

(i.) $3x^2+4xy-6y^2=0$.
(ii.) $2x^2-5xy+3y^2=0$.
(iii.) $4x^2-7xy-y^2=0$.

Find the slopes of the lines of the first equation.

9. Find the acute angles between the following pairs of lines.

(i.) $3x^2+4xy-2y^2=0$.
(ii.) $4x^2-8xy+3y^2=0$.

10. Obtain the acute angle between the lines $6x^2-7xy-20y^2=0$ and verify the result by finding the slope of each line from its equation.

11. Find the angles between the following pairs of lines.

(i.) $x^2-6xy+9y^2=0$,
(ii.) $4x^2+7xy-4y^2=0$.

What do you conclude in each case ?

12. Find the median centre of the triangle formed by the lines $8x^2-26xy+15y^2=0$ and $x+y=7$.

13. Find the orthocentre of the triangle formed by the lines

$$5x^2-7xy-6y^2=0$$
$$\text{and } 3x+7y=26.$$

14. Work out the combined equation to the straight lines through the points of intersection of the circle $x^2+y^2=9$ with the straight line $3x+2y=1$.

15. Find the acute angle between the pair of straight lines drawn from the origin to the intersections of the straight line $5x-2y+1=0$ with the circle $x^2+y^2=12$.

16. Find the acute angle between the pair of lines which join the origin to the intersections of the straight line $2x+y=20$ and the circle $x^2+y^2=85$.

17. Find the condition that the line $lx+my=1$ should touch the circle $x^2+y^2=8$ (Art. VIII. Ex. 3).

18. Find the condition that the chord cut off from the line $lx+my=1$ by the circle $x^2+y^2=5$ subtend a right angle at the centre.

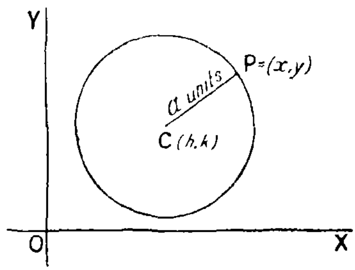
19. Use the method of Example 4, Art. VIII., to find the locus of the intersection of tangents to the circle $x^2+y^2=20$ which are at right angles to one another.

CHAPTER X

THE GENERAL EQUATION TO A CIRCLE

I. *Find the locus of a point which moves at a distance of a units from the fixed point* (h, k).

Let $P \equiv (x, y)$ be any point on the locus, and let $C \equiv (h, k)$. Then $CP = a$.



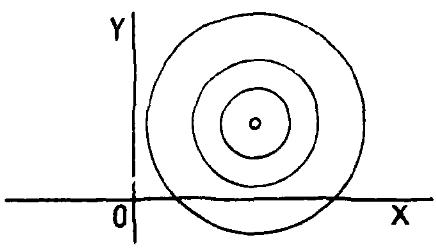
Now

$CP^2 = (x-h)^2 + (y-k)^2$ (Chap. I.),

$\therefore (x-h)^2 + (y-k)^2 = a^2$ (1).

This is the equation to the locus of $P \equiv (x, y)$.

It is therefore the equation to the circle whose centre is $C \equiv (h, k)$ and whose radius is a units. The equation is easily written down by equating



the square of the radius to the square of the distance between the points (x, y) and (h, k).

Note.—If we assign to a various values we shall obtain a system of concentric circles. If we take continuously decreasing values, then the circles will become smaller and smaller, tending to the point C as a limit. (Second Fig. of this Art.)

Hence when $a=0$ we say we have a "point circle" at (h, k). (Compare Chap. V. Art. I.)

If we square out, and collect terms in equation (1), we have

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0.$$

In this result put $h=-g$, $k=-f$, and $h^2+k^2-a^2=c$,

$$\therefore\ x^2+y^2+2gx+2fy+c=0 \qquad . \qquad . \quad (2).$$

In this form the centre is at the point $(-g, -f)$ since $h=-g$ and $k=-f$.

We observe: (i.) The equation is of the second degree in x and y, but is not homogeneous.

(ii.) The coefficients of x^2 and y^2 are equal.

(iii.) The term in xy is absent, or in other words the coefficient of xy is zero.

These are the characteristics of the equation to every circle, as we shall prove in next article.

EXAMPLE 1.—*Write down the equation to the circle whose centre is at the point* (2, 5) *and whose radius is* 3 *units.*

In this example $h=2$, $k=5$, and $a=3$.

$$\therefore\ (x-2)^2+(y-5)^2=9 \qquad . \qquad . \qquad . \quad (1),$$

$$\therefore\ x^2+y^2-4x-10y+20=0 \qquad . \qquad . \quad (2).$$

Compare these results with the two forms of the equation in the main discussion.

EXAMPLE 2.—*The centre of a circle is at the point* $(-\frac{2}{3}, -\frac{3}{4})$ *and its radius is* 2 *units. Find its equation.*

In this example $h=-\frac{2}{3}$, $k=-\frac{3}{4}$, and $a=2$.

$$\therefore\ (x+\tfrac{2}{3})^2+(y+\tfrac{3}{4})^2=4 \qquad . \qquad . \qquad . \quad (1),$$

$$\therefore\ x^2+y^2+\tfrac{4}{3}x+\tfrac{3}{2}y-\tfrac{431}{144}=0,$$

$$\therefore\ 144x^2+144y^2+192x+216y-431=0 \qquad . \quad (2).$$

In this example we note that the coefficients of x^2 and y^2 are not 1 but 144.

We also point out that the two equations (1) and (2) correspond to the two general forms found in this article, namely,

$$(x-h)^2+(y-k)^2=a^2,$$

$$\text{and } x^2+y^2+2gx+2fy+c=0.$$

II. *The graph of the equation* $x^2+y^2+2gx+2fy+c=0$ *is a circle.*

We can write the equation thus

$$(x^2+2gx)+(y^2+2fy)=-c$$

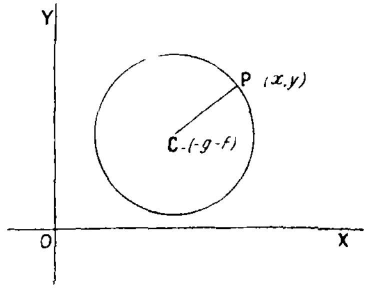
If we add g^2 to x^2+2gx and f^2 to y^2+2fy we make these expressions complete squares.

Add, then, g^2+f^2 to both sides of the equation.

$$\therefore\ (x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c,$$

$$\therefore\ (x+g)^2+(y+f)^2=g^2+f^2-c=\text{constant}.$$

This equation states that the square of the distance between the variable point (x, y) and the fixed point $(-g, -f)$ is constant.



Hence the distance between these points is constant.

The locus of (x, y) — that is, the graph of the equation $x^2+y^2+2gx+2fy+c=0$— is therefore a circle with centre

$$(-g, -f) \text{ and radius } \sqrt{g^2+f^2-c}.$$

Note that the co-ordinates of the centre are half the coefficients of x and y with the signs reversed.

Corollary.—$kx^2+ky^2+2gx+2fy+c=0$ is the equation to a circle.

Divide both sides by k,

$$\therefore\ x^2+y^2+\frac{2g}{k}x+\frac{2f}{k}y+\frac{c}{k}=0.$$

An equation such as this we have just shown to be that of a circle with its centre at the point $\left(-\frac{g}{k}, -\frac{f}{k}\right)$.

The form of the equation in the corollary arises when the co-ordinates of the centre are fractions.

Remark.—In all our theory we shall assume that the equation

to the circle has been arranged so as to have unity for the coefficients of x^2 and y^2.

EXAMPLE 1.—*Find the centre and radius of the circle*

$$x^2+y^2+6x-4y-3=0.$$

We have $(x^2+6x)+(y^2-4y)=3$,

$$\therefore (x^2+6x+9)+(y^2-4y+4)=9+4+3,$$
$$\therefore (x+3)^2+(y-2)^2=16.$$

Hence the point (x, y) moves at a distance of 4 units from the point $(-3, 2)$.

The centre of the circle is therefore at the point $(-3, 2)$, and its radius is 4 units.

EXAMPLE 2.—*Find the centre and radius of the circle*

$$x^2+y^2-2x-10=0.$$

We have $(x^2-2x)+y^2=10$,

$$\therefore (x^2-2x+1)+y^2=1+10,$$
$$\therefore (x-1)^2+(y-0)^2=11.$$

The centre is at the point $(1, 0)$ and the radius is $\sqrt{11}$ units.

EXAMPLE 3.—*The centre of a circle is at the point* $(3, 4)$ *and it passes through the origin, find its equation.*

(i.) Let a be the radius of the circle.

Then since $(3, 4)$ is its centre we have $(x-3)^2+(y-4)^2=a^2$.

But the origin lies on it,

$$\therefore 3^2+4^2=a^2,$$
$$\therefore (x-3)^2+(y-4)^2=25.$$

(ii.) *Aliter.*—If $C\equiv(3, 4)$ is the centre, then OC is a radius of the circle.

Now
$$OC^2=9+16$$
$$=25,$$
$$\therefore OC=5.$$

The equation to the circle is therefore

$$(x-3)^2+(y-4)^2=25 \text{ (Art. I.)},$$
$$\therefore x^2+y^2-6x-8y=0 \text{ as before.}$$

III. *Intersections of a straight line and a circle.*

The procedure is exactly the same as in the simpler case, when the origin was the centre of the circle.

Examples will make this clear.

EXAMPLE 1.—*Find the intersections of the straight line* $y=2x+1$ *and the circle* $x^2+y^2-3x+2y-4=0$.

Eliminate y,

$$\therefore x^2+(2x+1)^2-3x+2(2x+1)-4=0,$$
$$\therefore 5x^2+5x-1=0,$$
$$\therefore x=\cdot 17 \text{ or } -1\cdot 17 \text{ (nearly)}.$$

The values of y corresponding are obtained from the equation $y=2x+1$. They are

$$y=1{\cdot}34 \text{ or } -1{\cdot}34.$$

The points of intersection are $({\cdot}17,\ 1{\cdot}34)$ and $(-1{\cdot}17,\ -1{\cdot}34)$.

EXAMPLE 2.—*Show that the straight line $5x+2y=18$ touches the circle $x^2+y^2+x-6y+2=0$.*

Since $5x+2y=18$,

$$\therefore y=\frac{18-5x}{2}.$$

Substitute for y in the equation to the circle.

$$\therefore x^2+\frac{(18-5x)^2}{4}+x-\frac{6(18-5x)}{2}+2=0,$$

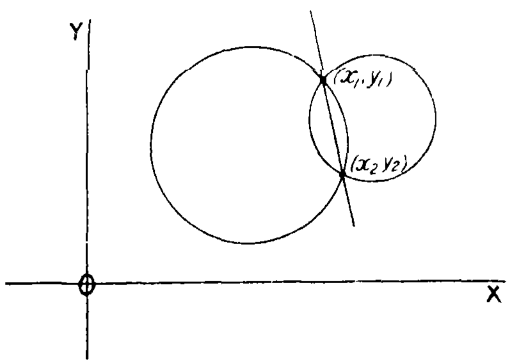
$$\therefore 29x^2-116x+116=0,$$

$$\therefore x^2-4x+4=0,$$

$$\therefore (x-2)^2=0.$$

Since both values of x are the same, it follows that the straight line cuts the circle in two coincident points. It is therefore a tangent.

IV. *The common chord, and points of intersection of two circles.*



Let the equations to two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0,$$

$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Let them intersect at the points (x_1, y_1) and (x_2, y_2)

Then since (x_1, y_1) is a point on both circles we have

$$x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1=0,$$

$$\text{and } x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2=0.$$

Subtract,

$$\therefore\ 2(g_1-g_2)x_1+2(f_1-f_2)y_1+(c_1-c_2)=0.$$

Similarly

$$2(g_1-g_2)x_2+2(f_1-f_2)y_2+(c_1-c_2)=0.$$

These two equations show us that the points (x_1, y_1) and (x_2, y_2) lie on the straight line $2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0$.

In other words, this is the equation to their join.

But since by hypothesis the points lie on the circles, therefore this is the equation to the common chord.

We remark that the equation of the common chord is found, simply by subtracting the equations to the two circles, it being understood that the coefficients of x^2 and y^2 are, or have been made, unity.

It should now be obvious how to find the co-ordinates of the points of intersection of two circles. The equation to the common chord having been found, it merely remains to find where this line cuts one of the circles. It will do so in two points real, coincident, or imaginary like any other straight line. It follows therefore that two circles intersect in two points real, coincident, or imaginary.

EXAMPLE 1.—*Find the equation to the common chord, and the points of intersection of the two circles* $x^2+y^2+x-4y-11=0$ and $x^2+y^2-11x-14y+3=0$.

(i.) Subtract the second equation from the first, and we have the equation to the common chord.

It is

$$12x+10y-14=0$$
$$\text{or } 6x+5y-7=0.$$

(ii.) To find where the circles intersect we shall find where the common chord cuts the first one.

From last equation we have

$$y=\frac{7-6x}{5}.$$

By substitution in the equation $x^2+y^2+x-4y-11=0$ we have

$$x^2+\frac{(7-6x)^2}{25}+x-\frac{4(7-6x)}{5}-11=0.$$
$$\therefore\ x^2+x-6=0,$$
$$\therefore\ x=2 \text{ or } -3.$$

From the equation to the common chord we have

$$y=-1 \text{ or } 5.$$

The points of intersection of the two circles are $(2, -1)$ and $(-3, 5)$.

EXAMPLE 2.—*Show that the circles* $x^2+y^2-8x-4y+10=0$ *and* $x^2+y^2-10x-10y+10=0$ *touch each other.*

On subtracting the two equations we find that the equation to their common chord is

$$2x+6y=0.$$

That is, $x=-3y$ . . . . (1).

Substitute for x in the equation

$$x^2+y^2-8x-4y+10=0,$$
$$\therefore 9y^2+y^2+24y-4y+10=0.$$

Whence $y^2+2y+1=0,$

$$\therefore (y+1)^2=0.$$

Hence both values of y are -1.

The line therefore cuts the circle in two coincident points.

It follows that the circles intersect in these two coincident points.

The equation to the common chord gives us $x=+3$, when $y=-1$.

The circles therefore touch at the point $(3, -1)$.

EXAMPLE 3.—*Prove that the circles* $x^2+y^2-3x+2y-1=0$ *and* $x^2+y^2-4x+y+3=0$ *have no real points of intersection.*

The equation to the common chord is

$$x+y-4=0,$$
$$\therefore y=4-x.$$

Substitute in the equation

$$x^2+y^2-3x+2y-1=0,$$
$$\therefore 2x^2-13x+23=0,$$
$$\therefore x=\frac{13\pm\sqrt{-15}}{4}.$$

The number under the radical being negative the values of x are unreal, hence the points of intersection are imaginary.

V. *Find the equation to the pair of lines joining the origin to the intersections of the line* $lx+my=1$ *and the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

Consider the equation

$$x^2+y^2+2(gx+fy)(lx+my)+c(lx+my)^2=0 \quad . \quad (1).$$

We see that it is formed by using the left-hand member of the linear equation to render all parts of the equation to the circle, homogeneous and of the second degree.

Now any point that satisfies both the equation to the line and the equation to the circle will satisfy last equation.

N

For let the line and the circle intersect at a point (x_1, y_1).

If this is a point on the last locus, then must

$$x_1^2 + y_1^2 + 2(gx_1 + fy_1)(lx_1 + my_1) + c(lx_1 + my_1)^2 = 0.$$

But $lx_1 + my_1 = 1$, so it follows that we must have

$$x_1^2 + y_1^2 + 2(gx_1 + fy_1) + c = 0,$$

which is true since (x_1, y_1) lies on the circle.

Simplify equation (1) and collect terms,

$$\therefore\ (1 + 2gl + cl^2)x^2 + 2(gm + lf + clm)xy + (1 + 2fm + cm^2)y^2 = 0.$$

This equation is homogeneous and of the second degree in x and y.

It is therefore the equation to a pair of straight lines passing through O.

We have already shown that they pass through the intersections of the straight line and the circle.

No more than the principle of this article should be remembered.

EXAMPLE.—*Find the equation to the pair of lines joining the origin to the intersections of the straight line $3x - 2y = 2$ and the circle $x^2 + y^2 - 6x - 4y - 40 = 0$.*

We have
$$\frac{3x - 2y}{2} = 1,$$

$$\therefore\ x^2 + y^2 - 2(3x + 2y)\frac{(3x - 2y)}{2} - 40\frac{(3x - 2y)^2}{4} = 0,$$

$$\therefore\ 4(x^2 + y^2) - 4(9x^2 - 4y^2) - 40(3x - 2y)^2 = 0,$$

whence $98x^2 - 120xy + 35y^2 = 0$.

VI. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*Find the equation to the circle which passes through the three points* (6, 0), (− 4, 0), *and* (0, 8).

Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Then since the points (6, 0), (− 4, 0), and (0, 8) lie on it we have

$$36 + 12g + c = 0 \quad . \quad . \quad . \quad (1),$$

$$16 - 8g + c = 0 \quad . \quad . \quad . \quad (2),$$

$$64 + 16f + c = 0 \quad . \quad . \quad . \quad (3).$$

On solving equations (1) and (2) we have

$$g = -1 \text{ and } c = -24.$$

Equation (3) then gives us $f = -\frac{5}{2}$.

The equation to the circle is therefore

$$x^2+y^2-2x-5y-24=0.$$

EXAMPLE 2.—*Find the equation to a circle passing through three given points not in a straight line.*

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be three points not in a straight line.

Let $x^2+y^2+2gx+2fy+c=0$ be the equation to the circle passing through them.

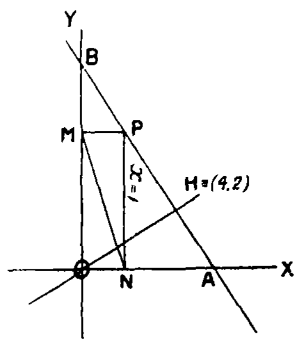
We notice that there are three constants in the equation to the circle, and we have to show how these are determined when three points are assigned.

Since the three points lie on the circle,

$$\therefore x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad . \quad . \quad (1),$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \quad . \quad . \quad (2),$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \quad . \quad . \quad (3).$$

We now have three linear equations in g, f, and c, which are therefore uniquely determined. (Compare Example 1.)

EXAMPLE 3.—*Through the point H (4, 2) a line is drawn perpendicular to OH, cutting OX and OY at A and B respectively. The line $x=1$ cuts AB at P. PN is perpendicular to OX and PM to OY. Prove that MN is a common tangent to the circles through H, M, and B, and H, N, and A.*



The gradient of OH is $\frac{2}{4}$, therefore that of AB is $-\frac{4}{2}$ or -2.

The equation to AB is therefore

$y-2=-2(x-4)$ (Chap. IX.),

$\therefore 2x+y=10$,

$\therefore A\equiv(5, 0)$ and $B\equiv(0, 10)$.

The line $x=1$ cuts $2x+y=10$ where $x=1$ and $y=8$.

Hence we have

$P\equiv(1, 8)$, $M\equiv(0, 8)$, $N\equiv(1, 0)$.

We must now find the equation to the circle through $H\equiv(4, 2)$, $M\equiv(0, 8)$, and $B\equiv(0, 10)$.

Let the equation be

$x^2+y^2+2gx+2fy+c=0$.

Then since (4, 2) lies on it,

$$\therefore 16+4+8g+4f+c=0,$$
$$\therefore 8g+4f+c=-20 \quad . \quad . \quad . \quad (1).$$

Similarly, since (0, 8) and (0, 10) lie on it,

$$16f+c=-64 \quad . \quad . \quad . \quad . \quad (2),$$
$$\text{and } 20f+c=-100 \quad . \quad . \quad . \quad . \quad (3).$$

On solving the last two equations we have

$$f=-9 \text{ and } c=80.$$

Equation (1) then gives us $g=-8$.

The equation to the circle is therefore

$$x^2+y^2-16x-18y+80=0 \quad . \quad . \quad . \quad (4).$$

Now the equation to MN is

$$\frac{x}{1}+\frac{y}{8}=1 \text{ (Chap. III.)}$$

$$\text{or } 8x+y=8 \quad . \quad . \quad . \quad . \quad (5).$$

We have to show that this line cuts the circle in two coincident points.

Equation (5) gives us

$$y=8-8x.$$

Substitute in equation (4).

$$\therefore x^2+(8-8x)^2-16x-18(8-8x)+70=0,$$

$$\therefore x^2=0,$$

$\therefore$ both values of x are zero.

Hence MN cuts the circle in two coincident points.

By equation (5) when $x=0$, $y=8$.

Thus the point of contact is (0, 8) or M.

Similarly, MN can be shown to touch the circle through H, N and A at N.

EXAMPLE 4.—*B is a fixed point on the y-axis. Through O and B lines are drawn at right angles to one another. Find the locus of their intersection.*

Let the lines intersect at $P\equiv(x, y)$.

Let $B\equiv(o, b)$.

The gradient of OP is $\dfrac{y}{x}$.

The gradient of BP is $\dfrac{y-b}{x}$.

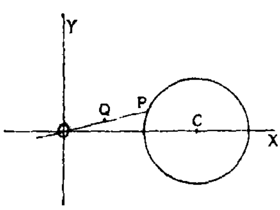
But OP and BP are at right angles,

$$\therefore \frac{y}{x}\times\frac{y-b}{x}=-1,$$

$$\therefore x^2+y^2-by=0.$$

The locus of P is a circle whose centre is at the mid-point of OB.

EXAMPLE 5.—*A point C is taken on the x-axis and a circle is described having this point as centre. Any point P is taken on the circumference. Find the locus of the mid-point of OP.*



Let Q be the mid-point of OP.

Let $P\equiv(x_1, y_1)$ and $Q\equiv(x', y')$.

Then

$$x_1=2x' \text{ and } y_1=2y' \quad . \quad (1).$$

Let $C\equiv(g, o)$.

Then the equation to the circle will be

$$+y^2-2gx+c=0 \text{ (Art. II.)}.$$

But P lies on the circle.

$$\therefore x_1^2+y_1^2-2gx_1+c=0 \quad (1)$$

$$\therefore (2x')^2+(2y')^2-2g(2x')+c=0 \text{ (by (1))},$$

which gives $4x'^2+4y'^2-4gx'+c=0$.

The locus of $Q\equiv(x', y')$ is the circle whose equation is

$$4x^2+4y^2-4gx+c=0.$$

Its centre is at the point $\left(\frac{g}{2}, o\right)$.

EXAMPLE 6.—*Find the equation to the circle on the join of $A\equiv(x_1, y_1)$ and $B\equiv(x_2, y_2)$ as diameter.*

Let $P\equiv(x, y)$ be any point on the locus.

Then the gradient of AP is $\frac{y-y_1}{x-x_1}$, and that of BP is $\frac{y-y_2}{x-x_2}$.

Now AP is perpendicular to BP by the geometry of a circle.

$$\therefore \frac{y-y_1}{x-x_1}\times\frac{y-y_2}{x-x_2}=-1,$$

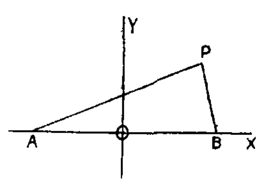
$\therefore (y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$ is the required equation.

If we multiply out and collect terms we have

$$x^2+y^2-(x_1+x_2)x-(y_1+y_2)y+x_1x_2+y_1y_2=0.$$

The method of this example should be remembered, but not the equation.

EXAMPLE 7.—*A and B are two fixed points. P moves so that $AP=3BP$. Find the locus of P.*



Take AB as axis of x and its right bisector as axis of y.

Let $AB=2a$ so that $A\equiv(-a, o)$ and $B\equiv(a, o)$.

Let $P\equiv(x, y)$.

Now $AP=3BP$.

$$\therefore AP^2=9BP^2,$$

$$\therefore (x+a)^2+y^2=9\{(x-a)^2+y^2\}.$$

Whence $8x^2+8y^2-20ax+8a^2=0$.

The locus of P is therefore the circle

$$2x^2+2y^2-5ax+2a^2=0.$$

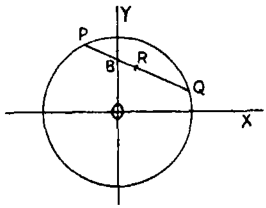
EXAMPLE 8.—*Chords of a circle are drawn through a fixed point B. Find the locus of their mid-points.*

Let O be the centre of the circle, a its radius.

Take OB as axis of y.

Let PQ be any chord which passes through B, and let R be its mid-point.

Let $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$, and $R\equiv(x', y')$.



The equation to PQ will be

$$y=mx+b \text{ (Chap. III.).}$$

The equation to the circle is

$$x^2+y^2=a^2.$$

We have as usual $x^2+(mx+b)^2=a^2$ by elimination of y.

$$\therefore (1+m^2)x^2+2bmx+b^2-a^2=0.$$

Now x_1 and x_2 are the roots of this equation,

$$\therefore x_1+x_2=-\frac{2bm}{1+m^2},$$

$$\therefore x'=\frac{x_1+x_2}{2}=-\frac{bm}{1+m^2} \quad . \quad . \quad . \quad (1).$$

But $y'=mx'+b$ since R is a point on PQ.

$$\therefore y'=-\frac{bm^2}{1+m^2}+b=\frac{b}{1+m^2} \quad . \quad . \quad . \quad (2).$$

Now m, the parameter of the pencil of chords, must not appear in the equation to the locus.

Results (1) and (2) give us

$$\frac{x'}{y'}=-m \quad . \quad . \quad . \quad . \quad (3).$$

Again, since $y'=mx'+b$,

$$\therefore m=\frac{y'-b}{x'} \quad . \quad . \quad . \quad . \quad (4),$$

$$\therefore \frac{y'-b}{x'}=-\frac{x'}{y'} \text{ (by 3 and 4),}$$

$$\therefore x'^2+y'^2-by'=0.$$

The locus of $R\equiv(x', y')$ is the circle $x^2+y^2-by=0$.

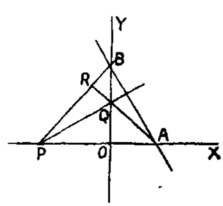
EXAMPLE 9.—*A straight line cuts XX' at A and YY' at B. Any line perpendicular to it cuts the x-axis at P and the y-axis at Q. Find the locus of the intersection of AQ and PB.*

Let $A\equiv(a, o)$, $B\equiv(o, b)$, and $R\equiv(x', y')$.

The equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

$\therefore -\frac{x}{b}+\frac{y}{a}=k$ is the equation to a line PQ perpendicular to it (Chap. III.).



$$\therefore P\equiv(-bk, o) \text{ and } Q\equiv(o, ak),$$

$$\therefore AQ\equiv\frac{x}{a}+\frac{y}{ak}=1 \text{ or } kx+y=ak,$$

$$\text{and } BP\equiv\frac{x}{-bk}+\frac{y}{b}=1 \text{ or } x-ky=-bk.$$

Now $R \equiv (x', y')$ lies on these lines,

$$\therefore\ kx' + y' = ak \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and } x' - ky' = -bk \quad . \quad . \quad . \quad (2).$$

Eliminate the line variable k.

By (1) we have $k = \dfrac{y'}{a - x'}$, and by (2) $k = \dfrac{x'}{y' - b}$,

$$\therefore\ \frac{y'}{a - x'} = \frac{x'}{y' - b},$$

which gives $x'^2 + y'^2 - ax' - by' = 0$.

The locus of $R \equiv (x', y')$ is the circle $x^2 + y^2 - ax - by = 0$.

RÉSUMÉ

1. The equation to the circle with centre (h, k) and radius a units is $(x - h)^2 + (y - k)^2 = a^2$.

Cor.—$(x - h)^2 + (y - k)^2 = 0$ is the equation to a point circle at (h, k).

2. $x^2 + y^2 + 2gx + 2fy + c = 0$ is the equation to a circle whose centre is at the point $(-g, -f)$.

3. The equation to the common chord of two circles is found by subtracting the equations of the circles, in which the coefficients of x^2 and y^2 must be unity.

4. The points of intersection of two circles are obtained by finding where the common chord cuts one of them.

EXAMPLES

1. Find the locus of a point which moves at a distance of 4 units from the point (5, 3).

2. Work out the equations of the following circles.

(i.) Centre (4, 2) : radius 3 units.
(ii.) Centre $(-6, 5)$: radius 2 units.
(iii.) Centre (0, 3) : radius 5 units.
(iv.) Centre $(-2, 0)$: radius 4 units.

3. Obtain *ab initio* as in Article II. the centres and radii of the following circles :

(i.) $x^2 + y^2 - 2x - 6y + 6 = 0$,
(ii.) $x^2 + y^2 + 4x - 8y + 11 = 0$,
(iii.) $x^2 + y^2 - 6x + 8y = 0$,
(iv.) $6x^2 + 6y^2 - 16x + 20y = 15$.

4. Find the area of the circles:

$$\text{(i.)}\ x^2+y^2-12x-5=0,$$
$$\text{(ii.)}\ 4x^2+4y^2-20x+24y+1=0.$$

5. Write down the equation to the system of concentric circles whose centre is the point $(-5, 2)$. Obtain the equation to the point member of the family.

6. Find the equation to the point member of the concentric system drawn round the point $(3, -1)$.

7. The centre of a circle is at the point $(-3, 4)$, and it passes through the origin. Find its equation.

8. Circles of fixed area are drawn so that all their circumferences contain a given point. Find the locus of their centres.

9. A circle of radius 5 units is drawn through the origin and has its centre at a distance of 4 units from YY' and on the positive side of it. Show that two circles can be drawn and find their equations.

10. Find the equation to the circle on the join of the points $(5, 2)$ and $(-1, 4)$ as diameter.

(Use the condition $m_1m_2=-1$ for perpendicular lines as in Ex. 6, Art. VI.)

11. Any line is drawn through the point $(0, 3)$ and a perpendicular is drawn to it from the origin. Find the locus of the foot of the perpendicular. (Art. VI., Ex. 4.)

12. AB is a given straight line. P is a variable point such that the square on AP is equal in area to $\Delta\, APB$. Prove that the locus of P is a circle.

13. CA and CB are two perpendicular lines of given length. P is a variable point such that the square on CP is equal in area to the quadrilateral $CAPB$. Prove the locus of P a circle.

14. A circle passes through two fixed points A and B and has its centre at a point C, on the right bisector of AB.

Take AB as axis of x and its right bisector as axis of y. Let $A\equiv(a, o)$, $B\equiv(-a, o)$, and $C\equiv(o, c)$. Work out the equation to the circle.

15. A point is taken on the x-axis and a system of concentric circles is drawn round it. Tangents are drawn from the origin to the members of the family. Prove that the locus of the points of contact is a circle.

(*Hint.*—Use the fact that the tangent is perpendicular to the corresponding radius.)

16. A circle is drawn with centre $(0, 5)$ and radius 2 units. P is a variable point on it. Q is taken on OP so that $OQ=3OP$. Find the locus of Q (Art. VI., Ex. 5).

17. Find the points of intersection of the straight line $y=2x-1$ with the circle $x^2+y^2-5x-3y+6=0$.

18. Find where the circle $x^2+y^2-4x+y=27$ cuts the straight line $3x-4y+17=0$.

What is the length of the chord cut off and where is its mid-point ?

19. Prove that the straight line $3x=y+13$ touches the circle $x^2+y^2-4x-6y+3=0$ and find the point of contact.

Draw the graphs.

20. Show that the line $8x+31y=100$ touches the circle

$$4x^2+4y^2+32x-y=0$$

and find the point of contact.

21. Show that the line $2x-3y+12=0$ does not cut the circle $x^2+y^2+6x+10y+20=0$.

22. The circle $x^2+y^2+2gx+2fy+c=0$ cuts XX' at two points A and B. Find the value of $OA \cdot OB$ and so obtain the length of the tangent from O to the circle.

23. Find the common chord and points of intersection of th circles $x^2+y^2+4x-12y+14=0$ and $x^2+y^2-14x+6y-22=0$.

24. Find the common chord and points of intersection of circles $x^2+y^2-14x-6y+32=0$ and $x^2+y^2+4x-18y+20=0$.

25. Show that the circles

$$x^2+y^2-12x-8y+42=0 \text{ and } 9x^2+9y^2+18x-30y-126=0$$

touch each other and find the point of contact.

26. Prove that the circles

$$x^2+y^2-12x-3y-18=0 \text{ and } x^2+y^2+4x+9y+18=0$$

touch at the point $(0, -3)$ and find the equation to the comn tangent.

27. In the last four examples find the equation to the line centres of each pair of circles.

Find also the distance between the centres.

28. Find the equation to the circum-circle of the triangle wh vertices are $(0, 0)$, $(1, 4)$, and $(3, 0)$.

29. Find the area of the circum-circle of the triangle wh vertices are the points $(0, -1)$, $(0, 2)$, $(3, 4)$.

Find also the equation to the diameter through the origin.

30. Prove that if a parallel be drawn through the centre of t circle $x^2+y^2-2fy=0$ to the x-axis, it will meet the bisector of $X\hat{O}$ on the circle.

31. Find the distance of the centre of the circle

$$x^2+y^2+2gx+2fy+c=0$$

from the line $lx+my=1$. If this distance is always constant wh is the locus of the centre ?

32. (i.) Find the condition that the circle $x^2+y^2+2gx+2fy+c=0$ should pass through the origin.

(ii.) Find the conditions that XX' should touch it at O.

(iii.) Deduce the locus of the centres of all circles which touch a given line at a given point.

33. Prove that the locus of the mid-points of chords of a circle which pass through a given point is a circle.

(*Hint.*—Take the centre as origin and the diameter through the given point as y-axis. See Ex. 8, Art. VI.)

34. A rod is free to turn in a vertical plane round a fixed peg. A lead bullet is suspended by a string fixed to the end of the rod.

Show that the path of the bullet as the rod turns is a circle.

35. A and B are two fixed points. P is a variable point such that $AP:PB=2:3$. Prove that the locus of P is a circle.

36. In last example show that the circle cuts the line AB harmonically at points C and D (see Chap. VII.).

37. Write down the equation to a circle whose centre is at the point $B \equiv (o, b)$ and which has BO as radius. Prove that it touches XX'.

38. O is a fixed point on the circumference of a circle, while P is a variable one. OP is produced its own length to Q. Prove that the locus of Q is a circle.

39. Any point P is taken on the circumference of a circle whose centre is at the point (2, 4) and whose radius is 3 units in length.

On OP a point Q is found such that $OQ =$ [illegible]OP. Prove that the locus of Q is a circle.

40. ABC is a triangle right angled at C. A variable line perpendicular to AB meets AC at P and BC at Q. Prove that the locus of the intersection of AQ and BP is the circum-circle of ΔABC. (See Ex. 9, Art. VI.)

41. A is a fixed point on a circle whose centre is C. AP is a variable chord of the circle. If Q is the mid-point of AP prove that its locus is a circle through A and C, touching the perpendicular to AC at A.

42. A is a fixed point on OX and B one on OY. C is another fixed point such that $OACB$ is a quadrilateral of area c sq. units. P is a variable point such that the square on OP is equal in area to the quadrilateral $ACBP$. Prove that the locus of P is a circle.

Derive a geometrical interpretation of the equation

$$x^2+y^2+2gx+2fy+c=0.$$

CHAPTER XI

TANGENTS: NORMALS: CHORDS OF CONTACT

THE analytical principles now to be applied to the general equation to a circle, are the same as in the simpler case, where the origin was the centre. Chapter VI. ought therefore to be revised.

I. *Find the gradient of the chord joining the points* (x_1, y_1) *and* (x_2, y_2) *on the circle* $x^2+y^2+2gx+2fy+c=0$.

The gradient of the line through the given points is

$$\frac{y_1-y_2}{x_1-x_2} \text{ (Chap. III.)} \quad . \quad . \quad . \quad (1).$$

Since the points lie on the circle,

$$\therefore x_1^2+y_1^2+2gx_1+2fy_1+c=0$$
$$\text{and } x_2^2+y_2^2+2gx_2+2fy_2+c=0.$$

Subtract,

$$\therefore (x_1^2-x_2^2)+(y_1^2-y_2^2)+2g(x_1-x_2)+2f(y_1-y_2)=0,$$
$$\therefore (x_1-x_2)(x_1+x_2+2g)+(y_1-y_2)(y_1+y_2+2f)=0,$$
$$\therefore (x_1-x_2)(x_1+x_2+2g)=-(y_1-y_2)(y_1+y_2+2f),$$
$$\therefore -\frac{x_1+x_2+2g}{y_1+y_2+2f}=\frac{y_1-y_2}{x_1-x_2} \quad . \quad . \quad . \quad . \quad . \quad (2).$$
$$=\text{gradient of the chord [by (1)].}$$

When finding the gradient of the chord joining two points on a circle the ordinary form (1) of the gradient can be used, but the form deduced in (2) has a particular advantage when we come to deal with the tangent, where the extremities of the chord coincide and we have to proceed to a limit.

EXAMPLE 1.—*Find the slope of the chord joining the points* (2, 3) *and* (−1, 4) *on the circle* $x^2+y^2-2x-10y+21=0$.

(i.) Use the expression for the gradient just found.

We have $g=-1$ and $f=-5$,

$$\therefore \tan\psi = -\frac{x_1+x_2+2g}{y_1+y_2+2f}$$
$$= -\frac{2-1-2}{3+4-10}$$
$$= -\tfrac{1}{3},$$
$$\therefore \psi = 161^\circ\ 34'.$$

(ii.) Use the ordinary expression for the gradient,

$$\therefore \tan\psi = \frac{y_1-y_2}{x_1-x_2}$$
$$= \frac{3-4}{2+1}$$
$$= -\tfrac{1}{3},$$
$$\therefore \psi = 161^\circ\ 34' \text{ as before.}$$

EXAMPLE 2.—*The line drawn from the centre of a circle to the mid-point of a chord, is perpendicular to the chord.*

Let the equation to the circle be $x^2+y^2+2gx+2fy+c=0$.

Then its centre is at the point $(-g, -f)$.

If (x_1, y_1) and (x_2, y_2) be the extremities of the chord, then its mid-point is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Hence the gradient of the line joining the centre to this point is by Chapter III. $\frac{\frac{1}{2}(y_1+y_2)+f}{\frac{1}{2}(x_1+x_2)+g}$, that is, $\frac{y_1+y_2+2f}{x_1+x_2+2g}$.

Now we have proved that the gradient of the chord is

$$-\frac{x_1+x_2+2g}{y_1+y_2+2f}.$$

The product of the gradients of the line and the chord is therefore −1.

It follows that the two lines are perpendicular (Chap. III.).

II. *Find the gradient of the tangent at the point* (x_1, y_1) *on the circle* $x^2+y^2+2gx+2fy+c=0$.

We saw in last article that the gradient of the chord joining two points (x_1, y_1) and (x_2, y_2) on the circle is $-\frac{x_1+x_2+2g}{y_1+y_2+2f}$.

In the case of the tangent these two points coincide.

$$\therefore x_2 = x_1 \text{ and } y_2 = y_1.$$

The gradient of the tangent is therefore $-\frac{2x_1+2g}{2y_1+2f}$ or $-\frac{x_1+g}{y_1+f}$.

EXAMPLE 1.—*Find the slope of the tangent at the point* $(-1, 4)$ *on the circle* $x^2+y^2-2x-10y+21=0$.

In this example $g=-1$ and $f=-5$.

Hence the gradient of the tangent at $(-1, 4) = -\dfrac{-1-1}{4-5} = -2$.

The slope is therefore $116^\circ\ 34'$.

EXAMPLE 2.—*At what points on the circle* $x^2+y^2-4x+6y-11{\cdot}2=0$ *is* $\cdot 5$ *the gradient of the tangent?*

Let (x_1, y_1) be such a point.

Then since $g=-2$ and $f=3$, the gradient of the tangent is

$$-\frac{x_1-2}{y_1+3},$$

$$\therefore\ -\frac{x_1-2}{y_1+3}=\tfrac{1}{2}\ \text{(by hypothesis)},$$

$$\therefore\ y_1=-2x_1+1 \quad . \quad . \quad . \quad (1).$$

But since (x_1, y_1) lies on the circle,

$$\therefore\ x_1^2+y_1^2-4x_1+6y_1-11{\cdot}2=0 \quad . \quad . \quad (2).$$

Hence by (1) and (2),

$$x_1^2+(-2x_1+1)^2-4x_1+6(-2x_1+1)-11{\cdot}2=0,$$

$$\therefore\ 5x_1^2-20x_1-4{\cdot}2=0,$$

$$\therefore\ x_1=4{\cdot}2 \text{ or } -{\cdot}2,$$

$$\therefore\ y_1=-19{\cdot}4 \text{ or } 1{\cdot}4 \text{ (by 1)}.$$

EXAMPLE 3.—*Prove that the line joining the centre of a circle to the point of contact of a tangent is at right angles to the tangent.*

Let $x^2+y^2+2gx+2fy+c=0$ be the circle.

Let (x_1, y_1) be the point of contact of a tangent.

Then since the centre of the circle is at the point $(-g, -f)$, it follows that the gradient of the line joining it to (x_1, y_1) is $\dfrac{y_1+f}{x_1+g}$.

But the gradient of the tangent is $-\dfrac{x_1+g}{y_1+f}$.

The product of the gradients is therefore -1, so that the lines are perpendicular.

III. *Find the equation to the tangent at the point* (x_1, y_1) *on the circle* $x^2+y^2+2gx+2fy+c=0$.

In Chapter VIII. it was shown that $y-y_1=m(x-x_1)$ is the equation to a line passing through the point (x_1, y_1).

Now in the case of the tangent

$$m=-\frac{x_1+g}{y_1+f}\ \text{(Art. II.)}.$$

The equation to the tangent is therefore

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1).$$

Whence $(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$,

$$\therefore\ (x_1 + g)x + (y_1 + f)y - x_1^2 - y_1^2 - gx_1 - fy_1 = 0.$$

But (x_1, y_1) lies on the circle.

$$\therefore\ x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Hence by addition

$$(x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0 \qquad . \quad (1).$$

This last equation can be written in a form easy to recall. Multiply out in equation (1) and collect terms as follows :

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \qquad . \quad (2).$$

Now if we express the terms in the equation to the circle in full the result is

$$xx + yy + g(x + x) + f(y + y) + c = 0.$$

In each term now replace one x or y by an x_1 or a y_1 as the case may be, giving

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

which is the equation (2) found above.

EXAMPLE 1.—*Find the equation to the tangent at the point* (3, 2) *on the circle* $x^2 + y^2 + 10x + 6y - 55 = 0$.

On writing the equation to the circle as directed above we have

$$xx + yy + 5(x + x) + 3(y + y) - 55 = 0.$$

Then on replacing one x by 3 and one y by 2 in the appropriate terms, we obtain the equation

$$3x + 2y + 5(x + 3) + 3(y + 2) - 55 = 0,$$
$$\therefore\ 8x + 5y - 34 = 0.$$

EXAMPLE 2.—*Obtain the equation to the tangent at the point* (2, -5) *on the circle* $x^2 + y^2 - 7x - 9y - 60 = 0$.

As usual we write $xx + yy - \frac{7}{2}(x + x) - \frac{9}{2}(y + y) - 60 = 0$.

Then $2x - 5y - \frac{7}{2}(x + 2) - \frac{9}{2}(y - 5) - 60 = 0$,

$$\therefore\ 3x + 19y + 89 = 0.$$

EXAMPLE 3.—*Find the length of the perpendicular from the centre of the circle* $x^2 + y^2 - 4x - 8y + 10 = 0$ *to the tangent at the point* (3, 1) *and prove it equal to the radius.*

The equation to the tangent at the point (3, 1) is found by the usual process to be

$$x - 3y = 0.$$

The centre of the circle is at the point (2, 4).

Hence the length of the perpendicular from the centre to the tangent

$$= \frac{2-12}{\sqrt{1+9}} \text{ (Chap. V.)}$$

$$= \sqrt{10} \text{ (neglecting sign)} \quad . \quad . \quad (1).$$

As shown in last chapter the equation to the circle can be written

$$(x-2)^2+(y-4)^2=10.$$

The radius of the circle is therefore

$$\sqrt{10} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the perpendicular is equal to the radius.

IV. *Find the equation to the normal at the point* (x_1, y_1) *on the circle* $x^2+y^2+2gx+2fy+c=0$.

The normal to a curve at a point on it, is the perpendicular through the point to the tangent there, as has already been stated in Chapter VI.

Since the gradient of the tangent at the point (x_1, y_1) is $-\frac{x_1+g}{y_1+f}$, that of the normal at the same point is $\frac{y_1+f}{x_1+g}$.

The equation to a line through the point (x_1, y_1) is

$$y-y_1=m(x-x_1).$$

In the case of the normal $m=\frac{y_1+f}{x_1+g}$ as we have just shown.

$$\therefore y-y_1=\frac{y_1+f}{x_1+g}(x-x_1),$$

$$\therefore \frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+g}.$$

This is the equation to the normal.

EXAMPLE 1.—*Write down the equation to the normal at the point* (3, 2) *on the circle* $x^2+y^2+10x+6y-55=0$.

In this example $g=5$, $f=3$, $x_1=3$ and $y_1=2$.

The equation to the normal is therefore

$$\frac{y-2}{2+3}=\frac{x-3}{3+5},$$

$$\therefore 5x-8y+1=0.$$

EXAMPLE 2.—*Prove that the normal to a circle passes through the centre.*

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Its centre is at the point $(-g, -f)$.
The equation to the normal is

$$\frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+g}.$$

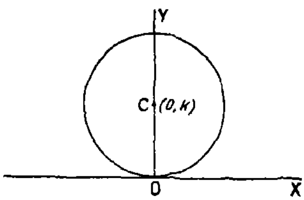
This line passes through the centre if

$$\frac{-f-y_1}{y_1+f}=\frac{-g-x_1}{x_1+g}.$$

That is, if $-1=-1$.
Hence the normal passes through the centre.

V. *Find the equation to a circle when the axes of reference are the tangent and normal at a point on it.*

We shall use the fact that the normal passes through the centre.

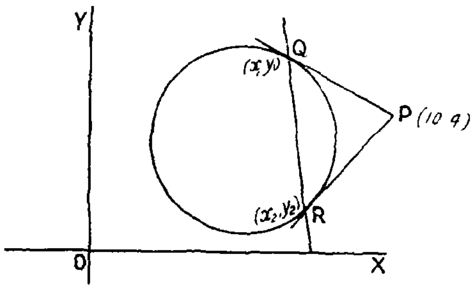


Let $C\equiv(o, k)$ be the centre of the circle, so that the radius is k units long.

Then the equation is

$$x^2+(y-k)^2=k^2,$$
$$\therefore\ x^2+y^2-2ky=0.$$

VI. EXAMPLE 1.—*A pair of tangents is drawn from the point* (10, 4) *to the circle* $x^2+y^2-2x-6y-3=0$. *Find the equation to their chord of contact.*



Let the tangents touch the circle at the points (x_1, y_1) and (x_2, y_2)
The equation to the tangent at the point (x_1, y_1) is

$$xx_1+yy_1-(x+x_1)-3(y+y_1)-3=0.$$

But the point (10, 4) lies on this line,

$$\because\ 10x_1+4y_1-(10+x_1)-3(4+y_1)-3=0.$$
$$\therefore\ 9x_1+y_1-25=0.$$

It follows that (x_1, y_1) lies on the line $9x+y-25=0$.

Similarly it can be shown that the point (x_2, y_2) lies on this line.

Hence $9x+y-25=0$ is the equation to the line which passes through the points (x_1, y_1) and (x_2, y_2), or in other words it is the equation to the chord of contact of the tangents from the point $(10, 4)$.

EXAMPLE 2.—*A pair of tangents is drawn from the origin to the circle $x^2+y^2+10x+18y+2=0$. Find the equation to the chord of contact.*

Let the points of contact be (x_1, y_1) and (x_2, y_2).

The equation to the tangent at the point (x_1, y_1) is

$$xx_1+yy_1+5(x+x_1)+9(y+y_1)+2=0.$$

Now this line passes through the origin,

$$\therefore 5x_1+9y_1+2=0$$

The point (x_1, y_1) therefore lies on the line $5x+9y+2=0$.

Similarly, it can be shown that the point (x_2, y_2) lies on this line.

Hence the equation is that of the line joining (x_1, y_1) and (x_2, y_2), that is to say, it is the equation to the chord of contact of the tangents from O to the circle.

VII. *Find the equation to the chord of contact of the pair of tangents drawn from the point (x', y') to the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

The method used in the two examples of last article is quite general, so that the reasoning applicable to the general case should be easy to follow.

Let (x_1, y_1) and (x_2, y_2) be the points of contact of the tangents from (x', y') to the circle.

The equation to the tangent at (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

But (x', y') lies on this line,

$$\therefore x'x_1+y'y_1+g(x'+x_1)+f(y'+y_1)+c=0,$$

$$\therefore (x'+g)x_1+(y'+f)y_1+gx'+fy'+c=0.$$

It follows that (x_1, y_1) lies on the line

$$(x'+g)x+(y'+f)y+gx'+fy'+c=0.$$

Similarly (x_2, y_2) is a point on this line.

Hence the equation is that of the line joining the points (x_1, y_1) and (x_2, y_2), or in other words it is the equation to the chord of contact of the tangents from (x', y') to the circle.

It can be arranged and written thus:

$$xx' + yy' + g(x + x') + f(y + y') + c = 0,$$

showing that its form is the same as that of a tangent.

The point (x', y') does not, however, satisfy the equation to the circle as it does not lie on the circumference.

The procedure followed in writing down the equation to a tangent is obviously employed in the case of the chord of contact.

EXAMPLE 1.—*Obtain the equation to the chord of contact of tangents drawn from the point $(-9, 2)$ to the circle $x^2+y^2-2x-2y+3=0$.*

As in the case of the tangent we write

$$xx+yy-(x+x)-(y+y)+3=0,$$

$$\therefore\ -9x+2y-(x-9)-(y+2)+3=0,$$

$$\therefore\ 10x-y=10.$$

EXAMPLE 2.—*Find the equation to the chord of contact of tangents drawn from O to the circle $x^2+y^2-2x-2y+3=0$.*

As before we write

$$xx+yy-(x+x)-(y+y)+3=0.$$

The required equation is therefore

$$-x-y+3=0$$

$$\text{or } x+y=3.$$

EXAMPLE 3.—*If (x_1, y_1) lies on the chord of contact of O with respect to the circle of last example, then O lies on the chord of contact of (x_1, y_1).*

We found that the chord of contact of tangents from O is $x+y=3$.

Since (x_1, y_1) lies on it,

$$\therefore\ x_1+y_1=3 \qquad (1).$$

Now the equation to the chord of contact of (x_1, y_1) is

$$xx_1+yy_1-(x+x_1)-(y+y_1)+3=0.$$

This line passes through the origin if $-x_1-y_1+3=0$, which is true in virtue of (1).

The chord of contact of (x_1, y_1) therefore passes through O.

VIII. *Find the length of the tangent drawn from a fixed point (x_1, y_1) to a given circle.*

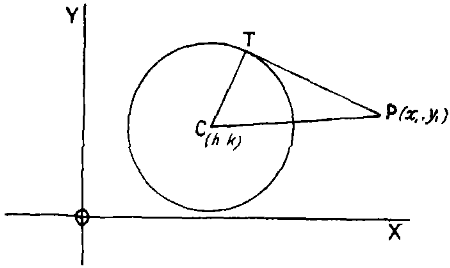
Let the radius of the circle be a units and let its centre be $C \equiv (h, k)$.

The equation to the circle is therefore $(x-h)^2+(y-k)^2=a^2$.

Now $PT^2 = CP^2 - CT^2$,

$$\therefore\ PT^2 = (x_1-h)^2+(y_1-k)^2-a^2,$$

$$\therefore\ PT = \sqrt{(x_1-h)^2+(y_1-k)^2-a^2}.$$



The expression for PT^2 is easily obtained as follows.

Arrange the equation to the circle thus:

$$(x-h)^2+(y-k)^2-a^2=0.$$

In the left-hand member put $x=x_1$ and $y=y_1$ and PT^2 is the result.

Corollary.—If the equation to the circle is

$$x^2+y^2+2gx+2fy+c=0,$$

then $$PT^2=x_1^{\,2}+y_1^{\,2}+2gx_1+2fy_1+c.$$

Note.—The equation to the circle must be arranged so as to have the coefficients of x^2 and y^2 unity.

EXAMPLE 1.—*Find the length of the tangent drawn from the point* (10, 3) *to the circle* $x^2+y^2-2x+4y-1=0$.

We have $$PT^2=100+9-20+12-1,$$
$$=100,$$
$$\therefore PT=10.$$

EXAMPLE 2.—*Find the length of the tangent drawn from the point* (2, 1) *to the circle* $x^2+y^2-2x+4y-12=0$.

In this case $$PT^2=4+1-4+4-12,$$
$$=-7,$$
$$\therefore PT=\sqrt{-7}.$$

Now $\sqrt{-7}$ is an imaginary number, hence no real tangent can be drawn from the point (2, 1) to the circle.

The point (2, 1) lies inside the circle.

EXAMPLE 3.—*Find the length of the tangent drawn from the origin to the circle of last example.*

We have $$OT^2=-12,$$
$$\therefore OT=\sqrt{-12}.$$

Hence OT is unreal, so that O also must be inside the circle.

IX. *The circle* $x^2+y^2+2gx+2fy+c=0$ *divides the plane of the axes into two regions, such that the expression*

$$x^2+y^2+2gx+2fy+c$$

has contrary signs for points in the two regions.

Let $P\equiv(x_1, y_1)$ be a point in the plane of the axes.

Let T be the point of contact of the tangent drawn from P (see figure, Art. VIII.).

Then $PT^2=x_1^2+y_1^2+2gx_1+2fy_1+c$.

Now if P is without the circle PT is real, so that PT^2 or $x_1^2+y_1^2+2gx_1+2fy_1+c$ must be positive.

If, however, P is within the circle PT is unreal, and the expression $x_1^2+y_1^2+2gx_1+2fy_1+c$ must be negative.

The last two examples of the previous article will have made this plain

EXAMPLE.—*Show that the origin and the point* (5, 3) *both lie outside the circle* $x^2+y^2-3x+7y+4=0$.

When $x=0$ and $y=0$ then $x^2+y^2-3x+7y+4=+4$.

When $x=5$ and $y=3$ then $x^2+y^2-3x+7y+4=+44$.

Both points therefore lie outside the circle, for real tangents can be drawn from them to the circle.

X. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*If from an external point a secant and a tangent be drawn to a circle, then the rectangle contained by the segments of the secant is equal to the square on the tangent.*

Take the external point as origin and the secant as axis of x.

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0,$$

and let the secant cut it at P and Q.

At the points where the secant cuts the circle $y=0$,

$$\therefore x^2+2gx+c=0.$$

If x_1 and x_2 are the roots of this equation, then

$$x_1x_2=c,$$

$$\therefore OP\cdot OQ=c,$$

$$=\text{square on the tangent from } O \text{ to the circle.}$$

EXAMPLE 2.—*Find the locus of the points of contact of the tangents drawn from a given point to a system of concentric circles.*

Take the given point as origin and the line joining it to the common centre C as axis of x.

Let $C\equiv(a, o)$.

Then $x^2+y^2-2ax+\lambda=0$ is the equation to a circle having $C\equiv(a, o)$ as centre. By varying λ we obtain the concentric system.

Let (x', y') be the point of contact of the tangent from O to a circle of the system,

$$\therefore x'^2+y'^2-2ax'+\lambda=0 \qquad . \qquad . \quad (1).$$

The equation to the tangent at (x', y') is

$$xx'+yy'-a(x+x')+\lambda=0.$$

But this line passes through the origin,

$$\therefore -ax'+\lambda=0,$$
$$\therefore \lambda=ax'.$$

Substitute in (1),

$$\therefore x'^2+y'^2-2ax'+ax'=0.$$

The locus of (x', y') is the circle $x^2+y^2-ax=0$.

EXAMPLE 3.—*A variable line is drawn through a fixed point and cuts a given circle at P and Q. Prove that the locus of the intersection of the tangents at P and Q is the chord of contact of the tangents from the fixed point.*

Let (h, k) be the fixed point.

Let the tangents at P and Q intersect at $R\equiv(x', y')$.

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

The equation to PQ the chord of contact of tangents from R is

$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

Now this line passes through the point (h, k),

$$\therefore hx'+ky'+g(h+x')+f(k+y')+c=0.$$
$$\therefore (h+g)x'+(k+f)y'+gh+fk+c=0.$$

The locus of (x', y') is the straight line

$$(h+g)x+(k+f)y+gh+fk+c=0.$$

It is the chord of contact of the tangents from the given point.

The equation may perhaps be more easily recognised if we re-arrange it as below,

$$hx+ky+g(x+h)+f(y+k)+c=0.$$

Note.—If the fixed point be chosen for origin the work is very much simplified.

RÉSUMÉ

1. The gradient of the chord joining the points (x_1, y_1) and (x_2, y_2) on the circle $x^2+y^2+2gx+2fy+c=0$ is $-\dfrac{x_1+x_2+2g}{y_1+y_2+2f}$.

2. The gradient of the tangent at the point (x_1, y_1) is $-\dfrac{x_1+g}{y_1+f}$.

3. The equation to the tangent at the point (x_1, y_1) on the same circle is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$
$$\text{or } (x_1+g)x+(y_1+f)y+gx_1+fy_1+c=0.$$

4. The equation to the normal at the same point is

$$\frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+f}.$$

5. The chord of contact of tangents from (x', y') to the circle is

$$xx'+yy'+g(x+x')+f(y+y')+c=0$$
$$\text{or } (x'+g)x+(y'+f)y+gx'+fy'+c=0.$$

6. The length of the tangent from (x', y') is

$$\sqrt{x'^2+y'^2+2gx'+2fy'+c}.$$

EXAMPLES

1. Prove that the points (2, 5) and (6, 3) lie on the circle

$$x^2+y^2-6x-4y+3=0.$$

Find the gradient of the chord joining them by using both the ordinary formula and that obtained in Article I.

2. In last question prove that the tangents at the points (2, 5) and (6, 3) are at right angles.

3. The points (−1, 6) and (3, 4) lie on the circle

$$x^2+y^2+8x+10y=89.$$

Find the gradients of the tangents at these points and of the chord of contact.

4. Find the gradient of the normal at the point (−2, 5) on the circle $x^2+y^2+3x-8y+17=0$.

5. At what points on the circle $x^2+y^2-4x+6y+3=0$ have the tangents a gradient of $\frac{1}{4}$.

6. Write down the equation to the tangent at the point (3, 4) on the circle $x^2+y^2+2x+6y-55=0$.

7. Write down the equations of the tangents in the following cases :—

(i.) point (2, 5) : circle $x^2+y^2-6x-4y+3=0$.
(ii.) point (−1, 6) : circle $x^2+y^2+8x+10y=89$,
(iii.) point (−2, 5) : circle $x^2+y^2+3x-8y+17=0$,
(iv.) point (0, 0) : circle $x^2+y^2+2gx+2fy=0$.

8. Find also the equations of the normals at these points, and in each case verify that they pass through the centre.

9. Find the area of the triangle made by the axes and the tangent to the circle $2x^2+2y^2-5x+6y-6=0$ at the point $(2, -4)$.

10. Write down the equations of the chords of contact of the tangents from the following points to the circles :—

(i.) point $(3, 4)$: circle $x^2+y^2+8x+5y+2=0$,
(ii.) point $(-2, 6)$: circle $x^2+y^2-4x+7y-5=0$,
(iii.) point $(5, -2)$: circle $3x^2+3y^2+27x-6y+12=0$,
(iv.) point $(0, 0)$: circle $5x^2+5y^2-15x-30y+16=0$,
(v.) point $(0, 0)$: circle $x^2+y^2+2gx+2fy+c=0$.

Work out the first example *ab initio*, giving the argument as in Article VI.

11. Prove that the chord of contact of tangents drawn from the point $(-4, 6)$ to the circle $x^2+y^2-6x-8y+12=0$ passes through the origin, and that the chord of contact of tangents from the origin passes through the point $(-4, 6)$.

12. A variable secant is drawn from the origin to cut the circle $x^2+y^2+5x-8y-6=0$ at the points P and Q.

Find the locus of R the intersection of the tangents to the circle at P and Q.

(*Hint.*—Let $R\equiv(x', y')$. Express the fact that its chord of contact passes through O.)

13. Calculate the length of the tangent from the point to the circle in each of the examples in Question 10.

14. If from an external point a secant and a tangent be drawn to a circle, prove that the rectangle contained by the segments of the secant is equal to the square on the tangent.

15. By finding the length of the tangent from O to the circle $x^2+y^2-7x+6y-8=0$, show that the origin is within the circle.

16. Tell whether the point is within or without the circle in each of the following cases :—

(i.) point $(5, 8)$: circle $x^2+y^2+6x+4y-4=0$,
(ii.) point $(-4, 3)$: circle $x^2+y^2+10x-5y-12=0$,
(iii.) point $(6, -1)$: circle $4x^2+4y^2-20x+18y+6=0$.

17. Distinguish between the two circles

$$x^2+y^2+2gx+2fy+c=0$$
$$\text{and } x^2+y^2+2gx+2fy-c=0.$$

18. Work out from a figure the equation to a circle which touches both OX and OY. Hence find two circles which touch the axes and pass through the point $(2, 4)$.

19. A point moves so that its distance from a fixed point is equal to the tangent from it to a given circle. Prove its locus to be a straight line.

20. A variable point is such that equal tangents can be drawn from it to two given circles. Show that its locus is a straight line.

21. The ratio of the tangents from a variable point to two given circles is constant. Prove that its locus is a circle.

22. A system of circles passes through a given point, and are such that the tangents drawn to them from a fixed point are of constant length. Prove that the locus of their centres is a straight line.

23. Find the locus of the mid-points of a system of parallel chords of

(i.) the circle $x^2+y^2+6x+10y+4=0$: gradient of chords $=\frac{1}{6}$,
(ii.) the circle $x^2+y^2+2gx+2fy+c=0$.

. (*Hint.*—Use the expression for the gradient found in Article I. taking (x', y') as the mid-point of the join of (x_1, y_1) and (x_2, y_2).

24. The common chord of the circles $x^2+y^2-2x-4y-20=0$ and $x^2+y^2-10x+4y-20=0$ meets them at A and B. Show that the four tangents at A and B are equidistant from the point $(2\frac{2}{3}, \frac{1}{3})$.

25. Find the equations to the pair of tangents which can be drawn from the point (2, 1) to the circle $x^2+y^2-2x+4y=0$.
(*Hint.*—Find where the chord of contact cuts the circle.)

26. AB is a chord of a circle, C the mid-point of one of the arcs cut off. Prove that C is equidistant from AB and the tangent at A.
Use the following steps:

(1) Take the mid-point of the arc as origin and the tangent and normal as axes. Hence obtain the equation to the circle. (See Art. V.)
(2) Let $A\equiv(x', k)$ where k is the distance of the mid-point from AB.
(3) Write down the equation to the tangent at A and find its distance from the origin. Then use in this result the fact that A lies on the circle.

27. If from an external point a pair of tangents at right angles to one another be drawn to a circle, then each tangent is equal to the radius of the circle.
Use this fact to prove that the locus of the external point is a circle taking as equation to the given circle

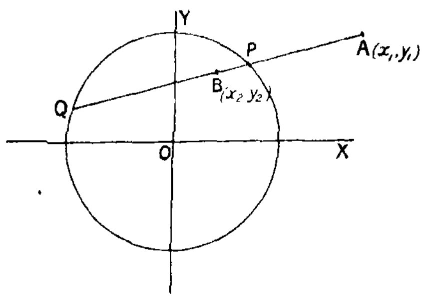
(i.) $x^2+y^2=a^2$
(ii.) $x^2+y^2+2gx+2fy+c=0$.

CHAPTER XII

CONJUGATE POINTS: POLES AND POLARS

The reader is recommended to revise now those portions of Chapter VII. which deal with harmonic division, and with the ratio in which the join of two points is cut by a circle.

I. *Definition of Conjugate Points with respect to a circle.*



Let A and B be two given points and let their join cut the circle in P and Q. We say that if P and Q harmonically separate A and B, then A and B are conjugate points with respect to the circle.

We must have $\dfrac{AP}{PB} = -\dfrac{AQ}{QB}$.

In other words, the circle cuts AB internally and externally in the same ratio.

II. *Find the condition that the points $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ may be conjugate with respect to the circle $x^2 + y^2 = a^2$.*

Let the circle cut the join of AB at a point (x', y') in the ratio $\lambda : 1$ (see figure, last article).

Then

$$x' = \frac{x_1 + \lambda x_2}{1 + \lambda} \text{ and } y' = \frac{y_1 + \lambda y_2}{1 + \lambda} \text{ (Chap. VIII.).}$$

Now (x', y') lies on the circle,

$$\therefore x'^2 + y'^2 = a^2,$$

$$\therefore \left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right)^2 + \left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right)^2 = a^2,$$

whence

$$(x_2^2 + y_2^2 - a^2)\lambda^2 + 2(x_1x_2 + y_1y_2 - a^2)\lambda + (x_1^2 + y_1^2 - a^2) = 0.$$

This equation is known as Joachimsthal's Ratio Equation.

It gives the values of the ratios $\dfrac{AP}{PB}$ and $\dfrac{AQ}{QB}$.

If these ratios are equal in value and opposite in sign then the roots of the equation are equal in value and of contrary sign.

$$\therefore x_1x_2 + y_1y_2 - a^2 = 0 \text{ (Chap. VII. Art. IV.)},$$

$$\therefore x_1x_2 + y_1y_2 = a^2.$$

This then is the condition that the points A and B should be conjugate with respect to the circle.

EXAMPLE 1.—*The points* (1, 2) *and* (3, 5) *are conjugate points with respect to a circle having O as centre. Find the equation to the circle.*

Let the required equation be

$$x^2 + y^2 = a^2.$$

Then by the condition $x_1x_2 + y_1y_2 = a^2$ we have $(1 \times 3) + (2 \times 5) = a^2$,

$$\therefore a^2 = 13,$$

$$\therefore x^2 + y^2 = 13.$$

EXAMPLE 2.—*Find a point on the x-axis conjugate with* (3, 5) *to the circle* $x^2 + y^2 = 15$.

Let the required point be (x', o).

Then from the condition $x_1x_2 + y_1y_2 = a^2$ we have $(3x') + (2 \times 0) = 15$,

$$\therefore x' = 5, \text{ and } (x', o) \equiv (5, 0).$$

EXAMPLE 3.—*Find the locus of points conjugate with* (8, 5) *to the circle* $x^2 + y^2 = 6$.

Let (x', y') be conjugate with (8, 5),

$$\therefore 8x' + 5y' = 6.$$

The locus of (x', y') is the straight line $8x + 5y = 6$.

III. *Imaginary Points and Lines.*

When dealing with the intersections of lines and circles we found that the algebraic equations which arose on elimination of a variable were often such as had unreal roots. The result

was that we had imaginary numbers as the co-ordinates of the points of intersection. We then said that such a straight line cut the circle in imaginary points, language which was quite a natural parallel to the algebraic results. We therefore see that on a real straight line there are imaginary points. Take, for example, the equation $x+y=3$. When $x=\sqrt{-2}$ then $y=3-\sqrt{-2}$. Thus the imaginary point

$$(\sqrt{-2}, 3-\sqrt{-2})$$

lies on the graph of the equation. In a similar manner we can obtain any number of such points. It therefore follows that a real straight line may be defined by two imaginary points. Let us find, for example, the straight line joining the imaginary points

$$(\sqrt{-1}, 3-2\sqrt{-1}) \text{ and } (-\sqrt{-2}, 3+2\sqrt{-2}).$$

Suppose its equation to be $lx+my=1$.

$$\therefore l\sqrt{-1}+m(3-2\sqrt{-1})=1 \quad . \quad . \quad (1),$$

$$\text{and } -l\sqrt{-2}+m(3+2\sqrt{-2})=1 \quad . \quad . \quad (2).$$

Subtract,

$$\therefore l(\sqrt{-1}+\sqrt{-2})-2m(\sqrt{-1}+\sqrt{-2})=0,$$

$$\therefore l-2m=0,$$

$$\therefore l=2m \quad . \quad . \quad . \quad . \quad (3).$$

Substitute for l in (1).

$$\therefore 2m\sqrt{-1}+m(3-2\sqrt{-1})=1,$$

$$\therefore 3m=1,$$

$$\therefore m=\tfrac{1}{3}.$$

Hence by (3) $l=\frac{2}{3}$.

The required equation is therefore

$$\tfrac{2}{3}x+\tfrac{1}{3}y=1 \text{ or } 2x+y=3.$$

By the help of this equation we can now draw the real straight line which passes through the two imaginary points. Ordinary geometry would not enable us to do so. We require to call to our aid algebra, through which we can find the equation to the line, and so draw the graph.

Consider next the equation $y-3=m(x-2)$. Its graph is

a straight line through the point (2, 3). If m has the value $\sqrt{-5}$ so that $y-3=\sqrt{-5}(x-2)$ then the straight line is an unreal one. As we can assign to m any number of imaginary values, it follows that through a real point such as (2, 3) there pass an infinite number of imaginary straight lines. Now the point (2, 3) lies within the circle $x^2+y^2=16$, and therefore no real tangents can be drawn from it. What, then, would happen if we were to try to find the condition that the line $y-3=m(x-2)$ should touch the circle? Though the geometrical construction for the tangents fails, the algebraic work in no way breaks down.

We have $y=m(x-2)+3$,

$$\therefore x^2+\{m(x-2)+3\}^2=16,$$

which gives $(1+m^2)x^2-2m(2m-3)x+(4m^2-12m-7)=0.$

The roots of this equation are equal if

$$4m^2(2m-3)^2=4(1+m^2)(4m^2-12m-7),$$

that is, if

$$12m^2+12m+7=0,$$

$$\therefore m=\frac{-3\pm 2\sqrt{-3}}{6}.$$

The equation gives the gradients of the tangents through the point (2, 3). But the roots of the equation are unreal, hence the gradients being imaginary numbers, the tangents are unreal. Their equations are $y-3=\frac{-3+2\sqrt{-3}}{6}(x'-2)$ and $y-3=\frac{-3-2\sqrt{-3}}{6}(x-2)$. These tangents will touch the circle in two imaginary points, so the question which naturally arises is "What about the line which joins these imaginary points, the chord of contact of these unreal tangents?" If we worked out the co-ordinates of the points of contact and the equation to the line joining them we would find the latter to be $2x+3y=16$, so that the chord of contact is a real line, not a surprising result, as we have already shown that two imaginary points may define a real straight line. The root of the whole matter is this that in analytical geometry we are

geometrically interpreting algebraic results, and the reality or unreality of the lines and points will depend on the nature of these algebraic results. Euclidean geometry is the counterpart of arithmetic, analytical geometry that of algebra.

Definition.—The chord of contact of tangents from a point to a circle is called the "Polar" of the point. The point is called the "Pole" of the chord of contact or polar.

IV. *The locus of the conjugate point of* (x_1, y_1) *with respect to the circle* $x^2+y^2=a^2$ *is the polar of* (x_1, y_1).

Let (x_2, y_2) be any harmonic conjugate of (x_1, y_1) with respect to the circle $x^2+y^2=a^2$.

$$\therefore\ x_1x_2+y_1y_2=a^2.$$

The locus of (x_2, y_2) is the straight line $xx_1+yy_1=a^2$.

But this is the equation to the chord of contact of tangents from (x_1, y_1) to the circle (Chap. VI. Art. V.), which proves our theorem.

If the point lies on the circle then the equation is that of the tangent to the circle at (x_1, y_1).

Hence the polar of a point on the circle is the tangent thereat.

The tangents are imaginary if (x_1, y_1) lies within the circle, but for a real circle and real point the polar is always a real line.

If the point is on the circumference the two tangents are coincident, and consequently coincide with their chord of contact, which is then the tangent at the point.

EXAMPLE 1.—*Find the polar of the point* (1, 2) *with respect to the circle* $x^2+y^2=8$ *and draw it.*

The process of writing down the polar is the same as in the case of a tangent at a point on the circle.

The polar of (1, 2) is therefore the line $x+2y=8$ (Diagram 1).

EXAMPLE 2.—*Find the pole of the line* $2x+3y=5$ *with respect to the circle* $x^2+y^2=10$.

Let (x', y') be the pole of the line.

The equation to the polar of this point is

$$xx'+yy'=10.$$

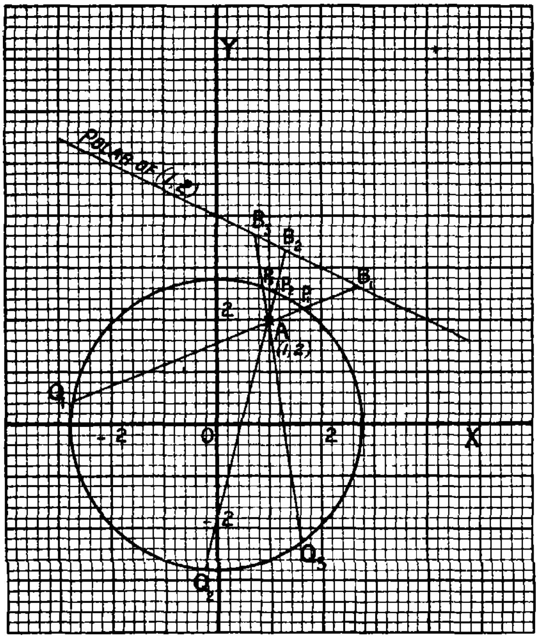


DIAGRAM 1.

Hence the lines $2x+3y=5$ and $xx'+yy'=10$ must be identical, and will therefore make the same intercepts on the axes. Equating these intercepts we have

$$\frac{5}{2}=\frac{10}{x'} \text{ (}x\text{-intercepts),}$$

$$\text{and } \frac{5}{3}=\frac{10}{y'} \text{ (}y\text{-intercepts),}$$

$$\therefore x'=4 \text{ and } y'=6,$$

$$\therefore (x', y')\equiv(4, 6).$$

EXAMPLE 3.—*Find where the polars of the points* (5, 2) *and* (7, 1) *with respect to the circle* $x^2+y^2=9$ *intersect. Prove that the point of intersection is the pole of the line joining the two given points.*

(i.) The polar of (5, 2) is $5x+2y=9$.

The polar of (7, 1) is $7x+y=9$.

On solving these equations we find that the polars intersect at the point (1, 2).

(ii.) The polar of the point (1, 2) is

$$x+2y=9 \quad . \quad . \quad . \quad . \quad (1).$$

The equation to the line joining (5, 2) and (7, 1) is

$$\frac{x-5}{7-5}=\frac{y-2}{1-2} \quad \text{(Chap. IX.)},$$

$$\therefore x+2y=9 \quad . \quad . \quad . \quad . \quad (2).$$

Hence by (1) and (2) the join of (5, 2) and (7, 1) is the polar of the intersection of their polars.

EXAMPLE 4.—*Find the pole of the line $lx+my+n=0$ with respect to the circle $x^2+y^2=a^2$.*

We may proceed as in Example 2, or as follows. Write the line in the polar form $xx_1+yy_1=a^2$.

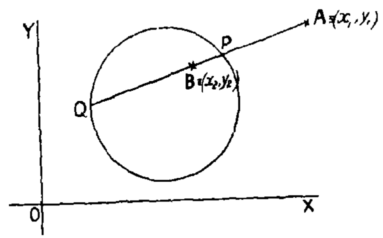
We have $lx+my=-n$,

$$\therefore -\frac{l}{n}x-\frac{m}{n}y=1,$$

$$\therefore -\frac{a^2l}{n}x-\frac{a^2m}{n}y=a^2.$$

This line is the polar of the point $\left(-\frac{a^2l}{n}, -\frac{a^2m}{n}\right)$.

V. *Find the condition that the points (x_1, y_1) and (x_2, y_2) be conjugate with respect to the circle $x^2+y^2+2gx+2fy+c=0$.*



Let the circle cut the join of (x_1, y_1) and (x_2, y_2) in the ratio $(\lambda : 1)$, at a point (x', y').

Then $x'=\frac{x_1+\lambda x_2}{1+\lambda}$ and $y'=\frac{y_1+\lambda y_2}{1+\lambda}$.

Now since (x', y') lies on the circle,

$$\therefore x'^2+y'^2+2gx'+2fy'+c=0,$$

$$\therefore \left(\frac{x_1+\lambda x_2}{1+\lambda}\right)^2+\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)^2+\frac{2g(x_1+\lambda x_2)}{1+\lambda}+\frac{2f(y_1+\lambda y_2)}{1+\lambda}+c=0,$$

$$\therefore\ (x_1+\lambda x_2)^2+(y_1+\lambda y_2)^2+2g(x_1+\lambda x_2)(1+\lambda)$$
$$+2f(y_1+\lambda y_2)(1+\lambda)+c(1+\lambda)^2=0,$$

whence

$$(x_2{}^2+y_2{}^2+2gx_2+2fy_2+c)\lambda^2$$
$$+2\{x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c\}\lambda$$
$$+(x_1{}^2+y_1{}^2+2gx_1+2fy_1+c)=0.$$

This is Joachimsthal's Ratio Equation for a circle whose centre is any point in the plane of the axes.

The roots of the equation give the ratios $\dfrac{AP}{PB}$ and $\dfrac{AQ}{QB}$.

Now if (x_1, y_1) and (x_2, y_2) are conjugate points these ratios are equal in value and of opposite sign (Art. I.).

Hence the roots of the equation are equal but of contrary sign, so that there can be no term in λ (Chap. VII. Art. IV.).

$$\therefore\ x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0.$$

This is the condition required. Writing it down is very similar to the process of writing out the equation to a tangent (Chap. XI.).

EXAMPLE 1.—*The centre of a circle is at the point* (4, 10) *and the points* (3, 4) *and* (7, 9) *are conjugate with respect to it. Find its equation.*

Its equation will be of the form

$$x^2+y^2-8x-20y+c=0.$$

If (3, 4) and (7, 9) are conjugate with respect to it, then by the condition found above we have

$$(3\times 7)+(4\times 9)-4(3+7)-10(4+9)+c=0,$$
$$\therefore\ c=93,$$
$$\therefore\ x^2+y^2-8x-20y+93=0.$$

EXAMPLE 2.—*A circle passes through the origin, the point* $(-1, 2)$, *and has the points* $(3, -2)$ *and* $(6, 5)$ *conjugate with respect to it. Find its equation.*

The equation to a circle passing through the origin is

$$x^2+y^2+2gx+2fy=0.$$

The point $(-1, 2)$ lies on it if

$$-2g+4f+5=0 \quad . \quad . \quad . \quad . \quad (1).$$

The points $(3, -2)$ and $(6, 5)$ are conjugate with respect to it if

$$(3\times 6)+(5\times -2)+g(3+6)+f(-2+5)=0,$$
$$\therefore\ 9g+3f+8=0 \quad . \quad . \quad . \quad . \quad (2).$$

Solving (1) and (2) gives $g = -\frac{17}{42}$ and $f = -\frac{61}{42}$,

$$\therefore x^2+y^2-\tfrac{17}{21}x-\tfrac{61}{21}y=0,$$

$$\therefore 21(x^2+y^2)-17x-61y=0.$$

Example 3.—*A circle passes through the origin, and the points* $(-1, -2)$ *and* $(5, 8)$ *are conjugate with respect to it. Find the locus of the centre.*

Let (x', y') be the centre.

The equation to the circle is therefore

$$x^2+y^2-2x'x-2y'y=0.$$

The points $(-1, -2)$ and $(5, 8)$ are conjugate with respect to it if

$$(-1\times5)+(-2\times8)-x'(-1+5)-y'(-2+8)=0,$$

$$\therefore 4x'+6y'+21=0.$$

The locus of the centre is the straight line $4x+6y+21=0$.

Example 4.—*Find the locus of the conjugate point of* $(3, 4)$ *with respect to the circle* $x^2+y^2-10x+8y-3=0$.

Let (x', y') be any point conjugate to $(3, 4)$,

$$\therefore 3x'+4y'-10(x'+3)+4(y'+4)-3=0,$$

$$\therefore 7x'-8y'+17=0.$$

The locus of (x', y') is the straight line $7x-8y+17=0$.

VI. *The locus of a point conjugate to* (x_1, y_1) *with respect to the circle* $x^2+y^2+2gx+2fy+c=0$ *is the polar of* (x_1, y_1).

Let (x_2, y_2) be any point conjugate to (x_1, y_1).

$$\therefore x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0.$$

The locus of (x_2, y_2) is the straight line

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

The equation is that of the chord of contact of tangents from (x_1, y_1) to the circle (Chap. XI. Art. VII.), which proves the theorem. Its equation is written down just as in the case of a tangent.

Example 1.—*Find the equation to the polar of the point* $(4, -3)$ *with respect to the circle* $x^2+y^2+8x-11y+2=0$.

The equation is

$$4x-3y+4(x+4)-\tfrac{11}{2}(y-3)+2=0,$$

$$\therefore 16x-17y+69=0.$$

Example 2.—*Find the polar of the origin with respect to the circle* $x^2+y^2+2gx+2fy+c=0$.

The equation is

$$ox+oy+g(x+o)+f(y+o)+c=0,$$

$$\therefore gx+fy+c=0.$$

VII. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*A straight line cuts the axes at P and Q so that $\frac{1}{OP}+\frac{1}{OQ}$ is constant. Find the locus of the pole of PQ with respect to the circle $x^2+y^2=a^2$.*

Let $\frac{1}{OP}+\frac{1}{OQ}=c$ (a constant).

Let (x', y') be the pole of PQ.

The equation to PQ is therefore

$$xx'+yy'=a^2,$$

$$\therefore OP=\frac{a^2}{x'} \text{ and } OQ=\frac{a^2}{y'}.$$

But $\frac{1}{OP}+\frac{1}{OQ}=c,$

$$\therefore \frac{x'}{a^2}+\frac{y'}{a^2}=c.$$

The locus of (x', y') is the straight line $x+y=a^2c$.

EXAMPLE 2.—*Find the locus of the poles, with respect to a given circle, of all straight lines passing through a fixed point.*

Take the point as origin and let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Let (x', y') be the pole of any line passing through O.

Now the polar of (x', y') is the line

$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

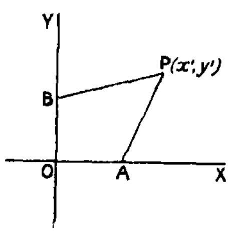
Since this line passes through the origin

$$\therefore gx'+fy'+c=0.$$

The locus of (x', y') is the straight line $gx+fy+c=0$.

It is the polar of the origin.

EXAMPLE 3.—*A and B are two fixed points on the axes. A variable point P is such that the quadrilateral $OAPB$ is constant in area. Prove that the polars of P with respect to the circle $x^2+y^2=c^2$ pass through a fixed point.*



Let $P\equiv(x', y')$, $A\equiv(a, o)$, and $B\equiv(o, b)$.

Then the polar of P is

$$xx'+yy'=c^2 \quad . \quad . \quad (1).$$

But $\quad \Delta OBP+\Delta OAP=\text{quad. } OAPB,$

$$=\frac{k}{2} \text{ (a constant)},$$

$$\therefore\ 2\Delta OBP + 2\Delta OAP = k,$$
$$\therefore\ bx' + ay' = k,$$
$$\therefore\ \frac{c^2}{k}(bx' + ay') = c^2 \qquad . \qquad . \qquad . \qquad (2).$$

Hence the equation to the polar of (x', y') is

$$xx' + yy' = \frac{c^2}{k}(bx' + ay') \ \{\text{by (1) and (2)}\},$$
$$\therefore\ (kx - bc^2)x' + (ky - ac^2)y' = 0,$$

which is the equation to a straight line passing through the intersection of the fixed straight lines $kx - bc^2 = 0$ and $ky - ac^2 = 0$, that is, through the fixed point $\left(\frac{bc^2}{k}, \frac{ac^2}{k}\right)$.

EXAMPLE 4.—*A point P is taken on the circle $x^2 + y^2 = a^2$ and perpendiculars PN and PM are drawn to the axes. If (x', y') is the pole of MN, prove* $\frac{1}{x'^2} + \frac{1}{y'^2} = \frac{1}{a^2}$.

Let $P \equiv (x_1, y_1)$. Then $N = (x_1, o)$ and $M = (o, y_1)$.
The equation to MN is therefore

$$\frac{x}{x_1} + \frac{y}{y_1} = 1.$$
$$\therefore\ \frac{a^2}{x_1}x + \frac{a^2}{y_1}y = a^2.$$

Now this last equation is the polar of $\left(\frac{a^2}{x_1}, \frac{a^2}{y_1}\right)$.
But MN is the polar of (x', y'),

$$\therefore\ x' = \frac{a^2}{x_1} \text{ and } y' = \frac{a^2}{y_1},$$
$$\therefore\ x_1 = \frac{a^2}{x'} \text{ and } y_1 = \frac{a^2}{y'}.$$

Now (x_1, y_1) lies on the circle,

$$\therefore\ x_1^2 + y_1^2 = a^2,$$
$$\therefore\ \frac{a^4}{x'^2} + \frac{a^4}{y'^2} = a^2,$$
$$\therefore\ \frac{1}{x'^2} + \frac{1}{y'^2} = \frac{1}{a^2}.$$

EXAMPLE 5.—*The radius of a circle is 3 units and the points (2, 4) and (7, 3) are conjugate with respect to the circle. Find the locus of the centre.*

Let (x', y') be the centre of the circle.
Then the equation to the circle is

$$(x - x')^2 + (y - y')^2 = 9,$$
$$\therefore\ x^2 + y^2 - 2x'x - 2y'y + (x'^2 + y'^2 - 9) = 0.$$

The points (2, 4) and (7, 3) are conjugate with respect to the circle,

$$\therefore 14+12-x'(2+7)-y'(4+3)+(x'^2+y'^2-9)=0,$$
$$\therefore x'^2+y'^2-9x'-7y'-17=0.$$

The locus of (x', y') is a circle having its centre at $(\frac{9}{2}, \frac{7}{2})$, the mid-point of the join of the conjugate points.

EXAMPLE 6.—*Find the equation to the pencil of lines which are the polars of points on YY' with respect to the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

(o, y') is any point on the y-axis.

Its polar is $yy'+gx+f(y+y')+c=0$,

$$\therefore gx+fy+c+y'(y+f)=0.$$

It is a straight line passing through the intersection of the line $y+f=0$, and of $gx+fy+c=0$ the polar of O.

As (o, y') moves along the y-axis we obtain a pencil of lines passing through the intersection of the pair of base lines.

EXAMPLE 7.—*From a variable point P perpendiculars PN and PM are drawn to the axes. If the polars of M and N with respect to the circle $x^2+y^2+2gx+2fy+c=0$ are at right angles, find the locus of P.*

Let $P\equiv(x', y')$,

$$\therefore N\equiv(x', o) \text{ and } M\equiv(o, y').$$

The polar of N is

$$xx'+g(x+x')+fy+c=0,$$
$$\text{or } (x'+g)x+fy+gx'+c=0 \quad . \quad . \quad . \quad (1).$$

Similarly the polar of M is

$$gx+(y'+f)y+fy'+c=0 \quad . \quad . \quad . \quad (2).$$

These lines are perpendicular if

$$(x'+g)g+(y'+f)f=0 \quad \{\text{Chap. III. Art. 1.}\},$$
$$\therefore gx'+fy'+g^2+f^2=0.$$

The locus of $P\equiv(x', y')$ is the straight line $gx+fy+g^2+f^2$.

It is parallel to the polar of O.

RÉSUMÉ

1. The points (x_1, y_1) and (x_2, y_2) are conjugate with respect to the circle $x^2+y^2=a^2$ if $x_1x_2+y_1y_2=a^2$.

2. The polar of the point (x_1, y_1) with respect to the above circle is the locus of all points conjugate to (x_1, y_1). Its equation is $xx_1+yy_1=a^2$.

3. The points (x_1, y_1) and (x_2, y_2) are conjugate with respect to the circle $x^2+y^2+2gx+2fy+c=0$

$$\text{if } x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0.$$

4. The polar of (x_1, y_1) with respect to this last circle is the line $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

5. The polar of a point is identical with the chord of contact of tangents drawn from the point.

EXAMPLES

1. Prove from definition that the points (3, 2) and (1, 1) are conjugate with respect to the circle $x^2+y^2=5$.

2. Use the test $x_1x_2+y_1y_2-a^2$ to prove that the following pairs of points are conjugate with respect to the circles.

(i.) points (2, 1) and (4, 4): circle $x^2+y^2=12$,
(ii.) points $(-3, 2)$ and $(-2, 5)$: circle $x^2+y^2=16$,
(iii.) points $(3, -6)$ and $(2, -2)$: circle $x^2+y^2=18$.

3. The points (3, 4) and (5, 2) are conjugate with respect to a circle having O as centre. Obtain its equation. (Art. II. Ex. 1.)

4. Obtain the equation to the circle which has O as centre and $(-6, 3)$ and $(-4, 2)$ as conjugate points.

5. Find a point on the line $5x+3y=30$ conjugate to the point (1, 3) with respect to the circle $x^2+y^2=12$.

6. The points (x, y) and (3, 4) are conjugate with respect to the circle $x^2+y^2=12$. Find and draw the locus of (x, y).

7. Write down the equations to the polars of the following points with respect to the given circles.—

(i.) point (2, 4): circle $x^2+y^2=7$,
(ii.) point $(-3, 5)$: circle $x^2+y^2=9$.
(iii.) point $(0, -4)$: circle $x^2+y^2=12$.
(iv.) point $(-3, -1)$: circle $3x^2+3y^2=8$.

8. Find the area of the triangle made with the axes by the polar of the point $(-5, 2)$ with respect to the circle $x^2+y^2=10$.

9. (x_1, y_1) and (x_2, y_2) are two points in the plane of the axes.
Write down the equations of their polars with respect to the circle $x^2+y^2=a^2$.
Work out the equation to a line through the origin and the intersection of the polars. Prove that it is perpendicular to the join of (x_1, y_1) and (x_2, y_2).

10. Find the pole of the line $2x-3y=4$ with respect to the circle $x^2+y^2=8$.

11. Find the pole of the line $5x=2y+6$ with respect to the circle $x^2+y^2=9$.

12. The polar of the point $P \equiv (x', y')$ with respect to the circle $x^2+y^2=a^2$ cuts the axes at Q and R.

Prove that the area of ΔOQR is $\frac{a^4}{2x'y'}$.

13. In last example draw perpendiculars PN and PM to the axes. Prove that if the area of ΔOQR is constant, then so is the area of the rectangle $ONPM$.

14. The equation to a system of concentric circles is $x^2+y^2=r^2$. Find the pole of the line $lx+my=1$ with respect to any one of them. (See Art. IV. Ex. 4.)
Use the results in next example.

15. Prove that the poles of a fixed straight line with respect to a system of concentric circles lie on a straight line through the common centre.

16. A pencil of lines has the point (h, k) as vertex.
Show that the locus of the poles of the rays, with respect to a circle having O as centre, is the polar of (h, k).
(*Hints.*—Take (x', y') as a pole. State the condition that its polar pass through the point (h, k).)

17. A and B are two fixed points. A perpendicular p_1 is drawn from A to the polar of B with respect to a circle having O as centre, and a perpendicular p_2 is drawn from B to the polar of A. Prove that $p_2 \times OA = p_1 \times OB$. (Salmon's Theorem.)

18. From points A and B tangents AP and BQ are drawn to a circle.
If $AP^2+BQ^2=AB^2$, prove that A and B are conjugate points with respect to the circle.

19. Prove that the following points are conjugate with respect to the given circles:—
(i.) points $(3, 5)$ and $(-8, 2)$: circle $x^2+y^2+4x+6y+3=0$,
(ii.) points $(-2, 0)$ and $(-6, -7)$: circle $x^2+y^2-8x+10y-9=0$,
(iii.) points $(0, 0)$ and $(-10, -2)$: circle $x^2+y^2+6x-18y+12=0$.

20. The points (x, y) and $(7, 5)$ are conjugate with respect to the circle $2x^2+2y^2-7x+9y-4=0$.
Find the locus of (x, y).

21. Write down the equations of the polars of the given points with respect to the circles.
(i.) point $(4, 2)$: circle $x^2+y^2+4x+10y+7=0$,
(ii.) point $(-6, 7)$: circle $x^2+y^2+6x-8y-9=0$,
(iii.) point $(0, -2)$: circle $x^2+y^2-5x+7y-2=0$,
(iv.) point $(0, 0)$: circle $x^2+y^2-8x+11y-3=0$,
(v.) point $(3, 2)$: circle $3x^2+3y^2+9x-4y+12=0$.

22. A variable straight line passes through the point $(3, -1)$.

(i.) Find the locus of its pole with respect to the circle
$$x^2+y^2-2x+3=0. \quad \text{(See Ex. 16.)}$$
(ii.) Show that it is the polar of $(3, -1)$.

23. A pencil of lines is drawn through the origin.
Find the locus of their poles with respect to the circle
$$x^2+y^2+4x-6y+5=0.$$

24. Prove that the polars of the point $(2, 1)$ with respect to the circles $x^2+y^2+4x-2y+5=0$ and $x^2+y^2-2x+6y+5=0$ intersect on the circle $x^2+y^2=5$.

25. A circle passes through the origin and the point $(4, 2)$. If the points $(5, -3)$ and $(2, -7)$ are conjugate with respect to it find its equation.

26. The pairs of points $[(3, 4)$; $(6, -2)]$ and $[(-2, 6)$; $(-3, -4)]$ are conjugate with respect to a circle passing through the origin. Find the equation to the circle.

27. A circle passes through the origin, and the points $(-1, 3)$ and $(2, -4)$ are conjugate with respect to it. Find the locus of its centre.

28. A variable circle passes through a fixed point D. The points A and B are conjugate with respect to it. Prove that the locus of its centre is a straight line. (*Hint.*—Take D as origin.)

29. A system of circles touches a given line at a given point A. Prove that the polars of a fixed point B form a pencil of lines. (See Chap. XI. Art. V. and Chap. VIII.)

30. A circle with centre O cuts the axes at A and B. If the pole of AB lie on the line $3x+2y=20$, find the length of the radius.

31. A variable straight line cuts the axes at P and Q. Find the locus of its pole with respect to the circle $x^2+y^2=a^2$ if
$$\text{(i.)} \quad \frac{1}{OP}+\frac{1}{OQ}=k,$$
$$\text{(ii.)} \quad OP+OQ=k,$$
$$\text{(iii.)} \quad OP \cdot OQ=k.$$

32. A circle is drawn having O as centre.
P_1 and P_2 are conjugate points with respect to it.
P_1N_1 and P_2N_2 are perpendiculars to XX'.
P_1M_1 and P_2M_2 are perpendiculars to YY'.
N is taken on the x-axis and M on the y-axis, so that ON is a mean proportional to ON_1 and ON_2, and OM is a mean proportional to OM_1 and OM_2.
The rectangle $ONKM$ is completed.
Prove that K lies on the circle.

33. Find the equation of the circle with respect to which the following pairs of points are conjugate :
$[(1, 1)$ and $(3, 5)]$; $[(5, -9)$ and $(-2, 3)]$; $[(0, -4)$ and $(4, 4)]$.

34. Find the equation to the pencil of lines which are the polars of points on XX' with respect to the circle $x^2+y^2+2gx+2fy+c=0$. (Art. VII. Ex. 6.)

35. Prove that the radius of the circle which has O as centre and (x_1, y_1) and (x_2, y_2) as conjugate points, is $\sqrt{x_1x_2+y_1y_2}$.

36. A circle passes through the points (a, o) and $(-a, o)$, and has the origin and (x', y') as conjugate points. Find its equation.

37. The centre of a circle is the point $(4, 6)$, and the points $(-1, 2)$ and $(3, -5)$ are conjugate with respect to it. Find the equation to the circle.

38. Prove that only one circle touches the axes and has the origin and a given point as conjugate points.
What is the length of its radius? (See Chap. XI. Ex. 25.)

39. The length of the tangent from the origin to a circle is 5 units, and the points $(3, 2)$ and $(5, -3)$ are conjugate with respect to it. Find the locus of the centre.

40. The radius of a circle is 3 units, and the points $(2, 4)$ and $(7, 3)$ are conjugate with respect to it. Find the locus of the centre.

41. If the points $(-2, 5)$ and $(4, -3)$ are conjugate with respect to a circle, and if the polar of the origin passes through the point $(1, 1)$ find the locus of the centre.

42. The polar of a point on the bisector of $X\hat{O}Y$ with respect to a circle of centre O cuts the axes at A and B. The rectangle $OAQB$ is completed. Prove that Q also lies on the bisector of $X\hat{O}Y$.

43. From a point P on a circle whose centre is O and radius a units long, perpendiculars PN and PM are drawn to the axes. If (x', y') is the pole of MN, prove that $\frac{1}{x'^2}+\frac{1}{y'^2}=\frac{1}{a^2}$.

44. From a variable point P perpendiculars PM and PN are drawn to two straight lines at right angles to one another.
The polar of M with respect to a given circle is at right angles to the polar of N. Prove that the locus of P is a straight line through the centre of the circle.
Show that this locus is the same for all members of a concentric system.

45. From a given point C perpendiculars CA and CB are drawn to two fixed lines at right angles to one another.
Prove that the locus of the centres of all circles with respect to which the polars of A and B are perpendicular, is a circle passing through the intersection of the fixed lines.

46. S_1 and S_2 are two given circles. Find the locus of a point whose polars with respect to the circles are, (i.) parallel, (ii.) perpendicular.

47. Prove that if the polar of a point P with respect to a circle passes through Q, then the polar of Q passes through P.

48. A point moves on a fixed straight line. Prove that its polars with respect to a given circle are concurrent.

(*Hints.*—Either take the centre of the circle as the origin, or take the pole of the given line as origin.)

CHAPTER XIII

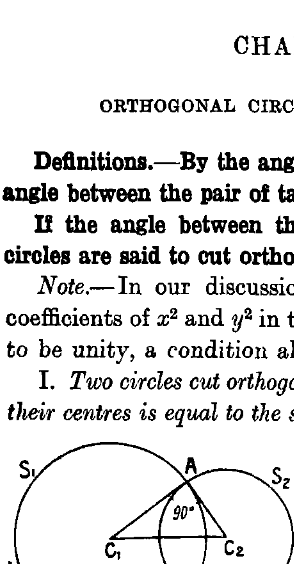
ORTHOGONAL CIRCLES AND COAXAL CIRCLES

Definitions.—By the angle between two circles is meant the angle between the pair of tangents at a point of intersection.

If the angle between the two tangents is a right angle the circles are said to cut orthogonally.

Note.—In our discussions, and in working examples, the coefficients of x^2 and y^2 in the equations to circles are supposed to be unity, a condition already postulated.

I. *Two circles cut orthogonally if the square on the line joining their centres is equal to the sum of the squares on their radii.*



AC_1 passes through the centre of S_1 since it is perpendicular to the tangent AC_2.

Similarly, AC_2 passes through the centre of S_2, AC_1 being a tangent to S_2.

Let C_1 and C_2 be the centres of the circles, and let r_1 and r_2 be their radii.

Then $\qquad C_1C_2^2 = AC_1^2 + AC_2^2$ (Pythagoras)

$$= r_1^2 + r_2^2$$

as was to be proved.

II. *Find the condition that the circles*

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{ and } x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

should cut orthogonally.

In this case $C_1 \equiv (-g_1, -f_1)$ and $C_2 \equiv (-g_2, -f_2)$.

Also $r_1^2 = g_1^2 + f_1^2 - c_1^2$ and $r_2^2 = g_2^2 + f_2^2 - c_2^2$ (Chap. X.).

Hence we have on stating that $C_1C_2^2 = r_1^2 + r_2^2$

$$(-g_1+g_2)^2+(-f_1+f_2)^2 = g_1^2+f_1^2-c_1^2+g_2^2+f_2^2-c_2^2$$

which gives $c_1+c_2=2(g_1g_2+f_1f_2)$.

In examples it is best to start from the condition

$$C_1C_2^2 = r_1^2+r_2^2,$$

but to save time we shall in some cases quote the condition just found.

EXAMPLE 1.—*Prove that the circles $x^2+y^2-3x+8y-2=0$ and $x^2+y^2+4x-5y-24=0$ cut orthogonally.*

We have
$$C_1C_2^2 = (\tfrac{3}{2}+2)^2+(-4-\tfrac{5}{2})^2$$
$$=\tfrac{109}{2},$$
$$\text{and } r_1^2+r_2^2 = (\tfrac{9}{4}+16+2)+(4+\tfrac{25}{4}+24)$$
$$=\tfrac{109}{2},$$
$$\therefore C_1C_2^2 = r_1^2+r_2^2.$$

Hence the circles cut orthogonally.

EXAMPLE 2.—*A variable circle passes through the origin and cuts the circle $x^2+y^2-2x+6y-8=0$ orthogonally. Find the locus of its centre.*

The centre of the fixed circle is $(1, -3)$.

Let (x', y') be the centre of the variable circle which passes through O. Its equation will therefore be $x^2+y^2-2x'x-2y'y=0$.

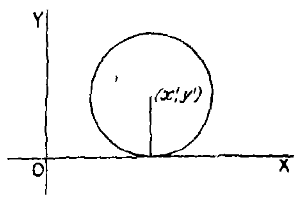
From the relation $C_1C_2^2=r_1^2+r_2^2$ we have

$$(x'-1)^2+(y'+3)^2=(1+9+8)+(x'^2+y'^2),$$
$$\therefore x'-3y'+4=0.$$

The locus of (x', y') is therefore the straight line $x-3y+4=0$.

EXAMPLE 3.—*A variable circle touches the x-axis and cuts the circle $x^2+y^2+4x+10y-2=0$ orthogonally. Find the locus of its centre.*

Let (x', y') be the centre of the variable circle.



From the figure we see that its radius is y'.

Its equation is therefore

$$(x-x')^2+(y-y')^2=y'^2,$$
$$\therefore x^2+y^2-2x'x-2y'y+x'^2=0.$$

This circle cuts the circle

$$x^2+y^2+4x+10y-2=0$$

orthogonally if $x'^2-2=2(-2x'-5y')$ (Art. II.: condition),

$$\therefore x'^2+4x'+10y'-2=0.$$

The locus of (x', y') is the graph of the equation

$$x^2+4x+10y-2=0.$$

It is called a parabola.

III. Definition.—The common chord of two circles is called their "Radical Axis."

IV. *Prove that the radical axis of two circles is the locus of points from which equal tangents can be drawn to the two circles.*

Let $P \equiv (x', y')$ be a point from which equal tangents can be drawn to the two circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Equating the squares of the tangents from P to both circles we have $x'^2+y'^2+2g_1x'+2f_1y'+c_1=x'^2+y'^2+2g_2x'+2f_2y'+c_2$,

$$\therefore 2(g_1-g_2)x'+2(f_1-f_2)y'+(c_1-c_2)=0.$$

The locus of P is the straight line

$$2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0.$$

Now this equation is evidently the same as is obtained by subtracting the equations of the two circles, hence it is that of their common chord or radical axis (Chap. X. Art. IV.).

The circles need not intersect in real points.

The nature of the intersections is a matter of indifference so far as the algebra is concerned.

EXAMPLE 1.—*Find the radical axis of the two circles*

$$5x^2+5y^2+15x-20y+18=0 \text{ and } 3x^2+3y^2-3x+9y-10=0.$$

We must write the equations to the circles with 1 as the coefficient of x^2 and y^2,

$$\therefore x^2+y^2+3x-4y+\tfrac{18}{5}=0$$
$$\text{and } x^2+y^2-x+3y-\tfrac{10}{3}=0.$$

Subtract,

$$\therefore 4x-7y+\tfrac{104}{15}=0,$$
$$\therefore 60x-105y+104=0.$$

This is therefore the equation to the radical axis.

EXAMPLE 2.—*Prove that the line joining the centres of two circles is perpendicular to their radical axis.*

Let the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Then the equation of their radical axis is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0.$$

The gradient of this line is

$$-\frac{g_1-g_2}{f_1-f_2} \quad . \quad . \quad . \quad . \quad (1).$$

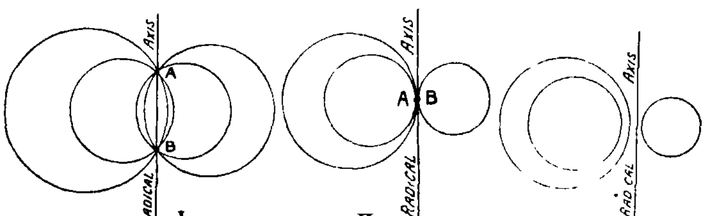
The centres of the circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$.
The gradient of their join is

$$\frac{-f_2+f_1}{-g_2+g_1} \text{ or } \frac{f_1-f_2}{g_1-g_2} \quad . \quad . \quad . \quad (2).$$

Hence by (1) and (2) the product of the gradients is -1.
The lines are therefore perpendicular.

V. *Coaxal Circles.—A system of circles is said to be coaxal when each of its members passes through two given points A and B.*

The name is due to the fact that the chord AB is common to every member of the system, so that each pair of circles has the same radical axis.



The points A and B may be real as in Fig. 1, coincident as in Fig. 2, and imaginary as in Fig. 3. It is all a matter of circles cutting each other in two points real, coincident, or imaginary. It is of no moment to the algebra of the geometry what the nature of the points are.

VI. *Equation to a circle passing through the intersections of two given circles.*

Let
$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Consider now the equation

$$(x^2+y^2+2g_1x+2f_1y+c_1)+\lambda(x^2+y^2+2g_2x+2f_2y+c_2)=0 \quad (1).$$

On collecting terms we have

$$(1+\lambda)x^2+(1+\lambda)y^2+2(g_1+\lambda g_2)x+2(f_1+\lambda f_2)y+c_1+\lambda c_2=0 \quad (2).$$

Form (2) shows us that the equation is that of a circle, for there is no term in xy, and the coefficients of x^2 and y^2 are equal, each being $(1+\lambda)$.

Equation (1) is satisfied by these values of x and y which make

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0$$

simultaneously.

It is therefore satisfied by the co-ordinates of the points of intersection of the two given circles.

Hence the equation is that of a circle, passing through the intersections of the two given circles.

Corollary 1.—*The equation we are discussing is that of a circle coaxal with the given pair, for all three pass through the same two points, namely the points of intersection of the given circles.*

Corollary 2.—*It is the equation to a coaxal system of circles, for we obtain an infinite number of circles all passing through the intersections of the given pair by merely varying* λ.

The two given circles are called the "base circles," and λ is called the "parameter" of the system.

Just as we spoke in Chapter VIII. of a pencil of lines passing through the intersection of a fixed pair, so can we now speak of a "pencil of circles" passing through the intersections of a pair of given circles.

When we put $\lambda=0$ in the equation

$$x^2+y^2+2g_1x+2f_1y+c_1+\lambda(x^2+y^2+2g_2x+2f_2y+c_2)=0$$

we have $$x^2+y^2+2g_1x+2f_1y+c_1=0.$$

If we write the equation thus

$$\frac{1}{\lambda}(x^2+y^2+2g_1x+2f_1y+c_1)+(x^2+y^2+2g_2x+2f_2y+c_2)=0,$$

and put $\lambda=\infty$, then, since $\frac{1}{\infty}=0$, we have

$$x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Thus 0 and ∞ are the values of the parameter which give the base circles of the system.

When $\lambda=-1$ we obviously obtain the equation to the radical axis. The radical axis is therefore a member of the system. Its radius is infinitely great. The reader will more readily

apprehend this if he draws a circle with a very large radius and observes how nearly straight any short arc is. As the radius is taken longer and longer, the circumference tends nearer and nearer to straightness.

Corollary 2.—*One and only one circle of a coaxal system passes through a given point.*

Every member of the system passes through two fixed points A and B. Hence if a third point C be given then only one member of the system can pass through it, since three points A, B and C uniquely define a circle.

The property also follows, from the fact that when the co-ordinates of a given point are substituted in the equation, then only one value of λ is obtained, so giving one circle. Example 1 below will illustrate this.

Corollary 3.—*The centres of a coaxal system are collinear.*

The centres of all the circles lie on the right bisector of the common chord.

Analytical proof : The equation to a coaxal system is

$$(x^2+y^2+2g_1x+2f_1y+c_1)+\lambda(x^2+y^2+2g_2x+2f_2y+c)=0,$$

which gives

$$x^2+y^2+\frac{2(g_1+\lambda g_2)}{1+\lambda}x+\frac{2(f_1+\lambda f_2)}{1+\lambda}y+\frac{c_1+\lambda c_2}{1+\lambda}=0.$$

Let (x, y) be the centre of any member of the system.

$$\therefore x=-\frac{(g_1+\lambda g_2)}{1+\lambda} \text{ and } y=-\frac{f_1+\lambda f_2}{1+\lambda}.$$

By eliminating λ we have $\dfrac{x+g_2}{x+g_1}=\dfrac{y+f_2}{y+f_1}$, whence on subtracting unity from both sides we obtain $\dfrac{x+g_1}{g_2-g_1}=\dfrac{y+f_1}{f_2-f_1}$ as the locus of (x, y). It is the straight line joining the centres $(-g_1, -f_1)$ and $(-g_2-f_2)$ of the base circles.

EXAMPLE 1.—*Find the equation to a circle passing through the origin and the intersections of the circles*

$$x^2+y^2=5 \textit{ and } x^2+y^2+3x-8y-10=0.$$

$$x^2+y^2+3x-8y-10+\lambda(x^2+y^2-5)=0$$

is the equation to a circle passing through the intersections of the given pair. If this circle passes through the origin

$$\text{then } -10-5\lambda=0,$$
$$\therefore \lambda=-2.$$

The required equation is therefore

$$x^2+y^2+3x-8y-10-2(x^2+y^2-5)=0$$
$$\text{or } x^2+y^2-3x+8y=0.$$

There is only one circle, which is consistent with the fact that three points define a circle.

Example 2.—*Find the equation to a circle passing through the origin, and the points where the line* $2x-3y=12$ *cuts the circle*

$$x^2+y^2=60.$$
$$x^2+y^2-60+\lambda(2x-3y-12)=0$$

is the equation to a circle passing through the intersections of the given line and circle. If it passes through the origin

$$\text{then } -60-12\lambda=0,$$
$$\therefore \lambda=-5.$$

The required equation is

$$x^2+y^2-60-5(2x-3y-12)=0$$
$$\text{or } x^2+y^2-10x+15y=0.$$

Example 3.—*Find the equation to a circle which touches the x-axis and is coaxal with the circles* $x^2+y^2+12x+8y-33=0$ *and* $x^2+y^2=5$.

$$x^2+y^2+12x+8y-33+\lambda(x^2+y^2-5)=0 \qquad . \quad (1)$$

is the equation to a circle coaxal with the given pair.
It cuts XX' or $y=0$ where $x^2+12x-33+\lambda(x^2-5)=0$,

$$\therefore (1+\lambda)x^2+12x-(33+5\lambda)=0.$$

Now the roots of this equation are equal, since XX' is to touch the circle,

$$\therefore 144=-4(1+\lambda)(33+5\lambda) \text{ (Chap. V. Art. VI.)},$$
$$\therefore 5\lambda^2+38\lambda+69=0,$$

which gives $\lambda=-3$ or $-\frac{23}{5}$.

There are thus two circles coaxal with the given pair, which touch the x-axis. This is in accordance with the fact that two circles can be drawn to pass through two given points and touch a given line.

From (1) their equations are

$$x^2+y^2-6x-4y+9=0$$
$$\text{and } 9x^2+9y^2-30x-20y-25=0$$

Example 4.—*Find the equation to a circle which touches the line* $x+2y=6$ *and passes through the intersections of the circle* $x^2+y^2=4$ *with the y-axis.*

$$x^2+y^2-4+\lambda x=0$$

is the equation to a circle passing through the intersections of the circle $x^2+y^2=4$ and the y-axis or $x=0$.

This circle has to touch the line $x+2y=6$.

Eliminating x to find the ordinates of their intersections we have, since $x=6-2y$,

$$\therefore (6-2y)^2+y^2+\lambda(6-2y)-4=0,$$
$$\therefore 5y^2-2(\lambda+12)y+2(3\lambda+16)=0.$$

The roots of this equation are equal since the points of intersection are coincident,

$$\therefore 4(\lambda+12)^2=40(3\lambda+16) \text{ (Chap. V. Art. VI.)},$$
$$\therefore \lambda^2-6\lambda-16=0,$$
$$\therefore \lambda=8 \text{ or } -2.$$

Hence two circles satisfy the given conditions, namely,

$$x^2+y^2-2x-4=0$$
$$\text{and } x^2+y^2+8x-4=0.$$

EXAMPLE 5.—*Discuss the coaxal system*

$$(x^2+y^2+2x-4)+\lambda(x^2+y^2-3x-4)=0$$

and draw some members of the system.

The base circles of the system are

$$x^2+y^2+2x-4=0$$
$$\text{and } x^2+y^2-3x-4=0.$$

Their radical axis is therefore $x=0$ or the y-axis.

This line cuts either circle at the points $(0, 2)$ and $(0, -2)$.

Hence all members of the system pass through these two points.

The equation

$$(x^2+y^2+2x-4)+\lambda(x^2+y^2-3x-4)=0$$

can be written

$$\frac{1}{\lambda}(x^2+y^2+2x-4)+(x^2+y^2-3x-4)=0.$$

Hence 0 and ∞ are the values of the parameter λ, which give the base circles. When $\lambda=-1$ we have the radical axis $x=0$. An infinity of unreal circles is obtained by assigning to λ imaginary values.

We show in Diagram 1 those members of the system for which $\lambda=0, \infty, -\frac{2}{7}, -\frac{1}{6}-6$, and -1.

VII. *Equation to a system of circles passing through the intersections of a given line and circle.*

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$ and that of the straight line be $lx+my+n=0$.

Then exactly the same reasoning as in the case of two circles will show that

$$x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0$$

is the equation to a circle passing through the intersections of

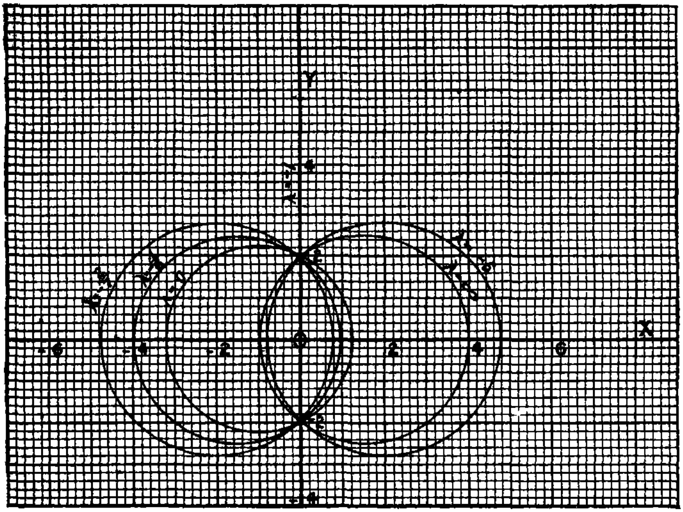


DIAGRAM 1.

the given line and circle (two points), and therefore that by varying λ a coaxal system is obtained.

Corollary.—$lx + my + n = 0$ is the radical axis of the system, since it joins the points common to all the members.

When we are free to choose the axes of reference ourselves, and to arrange other details in any problem connected with coaxal circles, we invariably take the line of centres as XX' and the radical axis is YY'. The system is then defined by means of YY' and a circle with O as centre, as we shall show in next article.

VIII. *Find in the most convenient form the equation to a system of coaxal circles.*

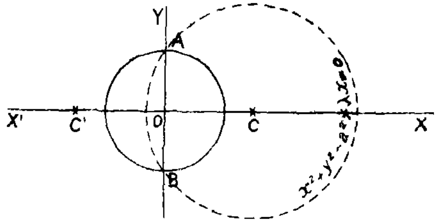
Suppose that A and B are the two fixed points of the system.

Take AB as axis of y and its right bisector as axis of x.

Describe a circle with centre O and radius OA. Let $OA = a$.

Then $x^2 + y = a^2$ is the equation to this circle.

Hence $x^2+y^2-a^2+\lambda x=0$ is the equation to a circle passing through the intersection of the circle just drawn and of the



line AB (Art. VII.). That is to say, it is a circle passing through A and B.

By varying λ a coaxal system is obtained, whose centres lie on XX', the right bisector of the common chord AB.

The equation $x^2+y^2-a^2+\lambda x=0$ is usually written in the form $x^2+y^2+2\lambda x+c=0$ for convenience.

The centre is at the point $(-\lambda, 0)$.

EXAMPLE 1.—*Prove that two members of a coaxal system have a given radius.*

We can easily see from the figure of the article that this will be the case. Suppose, for example, that the given radius is 5 units. Then a circle with centre A and radius 5 units will cut xx' at two points C and C', and either of the circles with centres C and C' and the given radius will pass through A and B. We proceed to prove this analytically.

Let the equation to the coaxal system be

$$x^2+y^2+2\lambda x+c=0.$$

Let r be the radius of any member.

Then $r^2=\lambda^2-c$,

$$\therefore \lambda=\pm\sqrt{r^2+c}.$$

Hence if r be given, then λ may have one of two values numerically equal but opposite in sign.

EXAMPLE 2.—*Find the members of the coaxal system*

$$x^2+y^2+2\lambda x+9=0$$

which have a radius of 4 units.

The square of the radius of any member $=\lambda^2-9$,

$$\therefore \lambda^2-9=16,$$

$$\therefore \lambda=\pm 5.$$

The circles are $x^2+y^2+10x+9=0$
and $x^2+y^2-10x+9=0$.

EXAMPLE 3.—*Prove that the polars of the origin with respect to the circles of the coaxal system of last example form a set of parallel straight lines.*

The polar of the origin with respect to any member of the above system is $\lambda x+9=0$ or $x=-\frac{9}{\lambda}$.

Hence for any given member we have a straight line parallel to YY', the radical axis.

EXAMPLE 4.—*In the system of Example 2, find the member which cuts orthogonally that circle for which* $\lambda=1$.

When $\lambda=1$ we have $x^2+y^2+2x+9=0$.

Let the member which cuts it orthogonally be $x^2+y^2+2\lambda x+9=0$.

The centres of these circles are $(-1, 0)$ and $(-\lambda, 0)$.

From the condition $C_1C_2{}^2=r_1{}^2+r_2{}^2$ we get

$$(\lambda-1)^2=(1-9)+(\lambda^2-9),$$

whence $\lambda=9$.

The required circle is therefore $x^2+y^2+18x+9=0$.

IX. *The Limiting Points of a Coaxal System.*

Definition.—The point circles of a coaxal system are called the "Limiting Points."

In Chapters V. and X. we stated that a point circle was one whose radius was zero (Art. I. of these chapters).

Since two circles of a coaxal system have a given radius it follows that there will be two which have zero radius, so that there will be two limiting points L and L'.

Let the equation to the coaxal system be

$$x^2+y^2+2\lambda x+c=0.$$

The square of the radius of any member $=\lambda^2-c$.

The radius is therefore zero when $\lambda=\pm\sqrt{c}$.

The equations to the point circles are therefore

$$x^2+y^2+2x\sqrt{c}+c=0$$
$$\text{and } x^2+y^2-2x\sqrt{c}+c=0.$$

These circles coincide with their centres, so that L and L' the limiting points are $(\sqrt{c}, o)$ and $(-\sqrt{c}, o)$. They are evidently imaginary if c is negative say $-a^2$. In that case

the common points A and B of the system are real, for the line $x=0$ cuts any circle of the system.

$$x^2+y^2+2\lambda x-a^2=0 \text{ (putting } c=-a^2\text{)},$$

where $y^2-a^2=0$,

that is, where $y=\pm a$ (real numbers).

EXAMPLE 1.—*Find the limiting points of the coaxal system*

$$x^2+y^2+2\lambda x+9=0.$$

The square of the radius of any member $=\lambda^2-9$.
The radius is therefore of zero length when $\lambda=\pm 3$.
Hence the point circles of the system are

$$x^2+y^2+6x+9=0$$
$$\text{and } x^2+y^2-6x+9=0.$$

These circles coincide with their centres, the limiting points.

$$\therefore L\equiv(-3, 0) \text{ and } L'\equiv(3, 0).$$

EXAMPLE 2.—*Show that the coaxal system* $x^2+y^2+2\lambda x-7=0$ *has imaginary limiting points.*

The square of the radius of any member $=\lambda^2+7$.

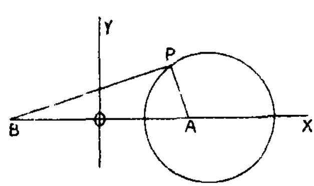
The radii of the point members are therefore given by $\lambda^2+7=0$ or $\lambda=\pm\sqrt{-7}$.

Since λ is unreal it follows that the limiting points are imaginary.

X. *The Circle of Apollonius.—A point P moves so that the ratio of its distances from two fixed points A and B is constant. Prove that the locus of P is a circle.*

Take AB as axis of x and its right bisector as axis of y.

Let $\dfrac{AP}{PB}=\sqrt{\lambda}$.



Let $P\equiv(x, y)$ and $A\equiv(a, o)$.

Then $B\equiv(-a, o)$.

Now, since $\dfrac{AP}{PB}=\sqrt{\lambda}$,

$$\therefore \frac{AP^2}{PB^2}=\lambda,$$

$$\therefore \frac{(x-a)^2+y^2}{(x+a)^2+y^2}=\lambda.$$

$$\therefore (1-\lambda)x^2+(1-\lambda)y^2-2a(1+\lambda)x+a^2(1-\lambda)=0,$$

$$\therefore x^2+y^2-\frac{2a(1+\lambda)}{1-\lambda}x+a^2=0.$$

The locus of $P \equiv (x, y)$ is a circle having $\left\{\frac{a(1+\lambda)}{1-\lambda}, o\right\}$ as centre. It is called the Circle of Apollonius.

Corollary.—If λ be varied from o to ∞ a system of coaxal circles having YY' as radical axis is obtained. We show them in the diagram below, taking $AB = 2$ units.

The limiting points can easily be shown to be A and B. Regarded as point circles they have as their parameters $\lambda = 0$ and $\lambda = \infty$.

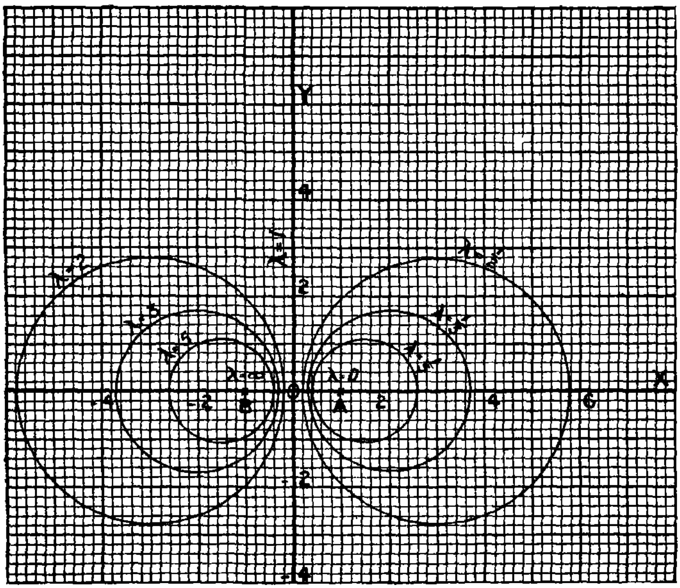


DIAGRAM 2.

XI. MISCELLANEOUS EXAMPLES

EXAMPLE 1.—*A variable circle cuts a given circle orthogonally. If the tangents from a fixed point to both circles are equal, find the locus of the centre of the variable circle.*

Take the fixed point as origin and let the equation to the given circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Let (x', y') be the centre of the variable circle whose equation will then be

$$x^2+y^2-2x'x-2y'y+c'=0.$$

The circles cut orthogonally if

$$c+c'=-2(gx'+fy') \text{ (Art. II.)} \quad . \quad (1).$$

But the tangents from O to both circles are equal, therefore by equating their squares we have

$$c=c' \quad . \quad . \quad . \quad . \quad (2).$$

Hence by (1) and (2),

$$2c=-2(gx'+fy'),$$

$$\therefore gx'+fy'+c=0.$$

The locus of (x', y') is the straight line $gx+fy+c=0$.

It is the polar of O with respect to the given circle.

EXAMPLE 2.—*A variable circle passes through a fixed point and cuts a given circle orthogonally. Find the locus of its centre.*

Take the fixed point as origin, and the line through it and the centre of the given circle as axis of x.

Let (x', y') be the centre of the variable circle.

Then the equation to the given circle will be

$$x^2+y^2+2gx+c=0,$$

and that of the variable circle which passes through O

$$x^2+y^2-2x'x-2y'y=0.$$

These circles cut orthogonally,

$$\therefore c=-2gx' \text{ (see Art. II.)}.$$

Hence the locus of (x', y') is the straight line $x=-\dfrac{c}{2g}$.

It is parallel to YY'.

EXAMPLE 3.—*In a system of coaxal circles prove that the polar of a limiting point is the same for all members of the system.*

Let the equation to the system be

$$x^2+y^2+2\lambda x+c^2=0.$$

Then square of radius of any member $=\lambda^2-c^2$.

This is zero when $\lambda=\pm c$.

The limiting points are therefore (c, o) and $(-c, o)$, as these are the centres of the point members of the system.

The polar of (c, o) with respect to any member of the system is

$$cx+\lambda(x+c)+c^2=0,$$

$$\therefore c(x+c)+\lambda(x+c)=0,$$

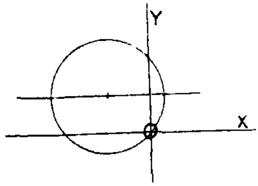
$$\therefore (x+c)(c+\lambda)=0.$$

Now $c+\lambda\neq 0$ in general,

$$\therefore x+c=0.$$

Hence the polar of (c, o) is the line parallel to the radical axis (*i.e.* YY'), which passes through the other limiting point $(-c, o)$.

EXAMPLE 4.—*Prove that all circles which pass through a fixed point and have their centres on a given line pass through another fixed point.*



Take the fixed point as origin and the perpendicular from it to the given line as axis of y.

Then the centres of all the circles have the same ordinate.

Let (x', f) be the centre of one of these circles.

Its equation will therefore be

$$x^2+y^2-2x'x-2fy=0.$$

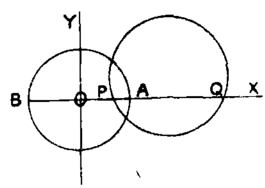
Now f is a constant as we have already indicated, but x' varies from circle to circle.

Hence the circles form a coaxal system passing through the intersections of the line $x=0$ with the circle $x^2+y^2-2fy=0$.

The line $x=0$ cuts this circle where $y^2-2fy=0$, that is, where $y=0$ or $2f$.

The second fixed point is therefore $(0, 2f)$.

EXAMPLE 5.—*Prove that if two circles cut orthogonally, then any diameter of the first is cut harmonically by the second.*



Take the diameter AB of the first as axis of x and let the equation to the circle be $x^2+y^2=a^2$.

Let the equation to the second circle be $x^2+y^2+2gx+2fy+c=0$.

Since the two circles cut orthogonally,

$$\therefore c-a^2=0 \text{ (Art. II.)},$$

$$\therefore c=a^2.$$

The equation to the second circle is therefore

$$x^2+y^2+2gx+2fy+a^2=0 \quad . \quad . \quad . \quad (1).$$

Let it cut AB the x-axis at a point P in the ratio $\lambda : 1$.

Now $A\equiv(a, o)$ and $B\equiv(-a, o)$,

$$\therefore P\equiv\left\{\frac{a(1-\lambda)}{1+\lambda},\ o\right\}.$$

Hence since P lies on (1),

$$\therefore \frac{a^2(1-\lambda)^2}{(1+\lambda)^2}+\frac{2ag(1-\lambda)}{1+\lambda}+a^2=0.$$

Which gives $(g-a)\lambda^2-(g+a)=0$

$$\therefore \lambda=\pm\sqrt{\frac{g+a}{g-a}}.$$

Since the two values of λ are of the same magnitude but of opposite sign, it follows that the second circle divides AB externally

and internally in the same ratio, or in other words divides it harmonically.

EXAMPLE 6.—*A variable circle cuts two fixed circles. Prove that the radical axes of the variable circle and each of the fixed circles intersect on the radical axis of the given pair.*

Take the line joining the centres of the fixed circles as axis of x, and their radical axis as axis of y.

Their equations are therefore

$$x^2+y^2+2\lambda_1 x+k=0 \quad . \quad . \quad . \quad (1)$$

$$\text{and } x^2+y^2+2\lambda_2 x+k=0 \quad . \quad . \quad (2).$$

Let the equation to the variable circle be

$$x^2+y^2+2gx+2fy+c=0 \quad . \quad . \quad . \quad (3).$$

The radical axes of (1) and (3) and of (2) and (3) are respectively

$$2(g-\lambda_1)x+2fy+c-k=0$$

$$\text{and } 2(g-\lambda_2)x+2fy+c-k=0.$$

By subtraction these lines intersect on the line $2(\lambda_2-\lambda_1)x=0$ or $x=0$ (Chap. VIII. Art. VI.), the radical axis of (1) and (2).

RÉSUMÉ

1. Two circles with centres C_1 and C_2 and radii r_1 and r_2 cut orthogonally if $C_1C_2^2=r_1^2+r_2^2$.

The analytical condition is $c_1+c_2=2(g_1g_2+f_1f_2)$.

2. The common chord of two circles is called their "Radical Axis." It is the locus of points from which equal tangents can be drawn to the two circles.

3. A coaxal system of circles is one whose members all pass through two fixed points. These can be real, coincident, or imaginary. The equation to such a system is

$$(x^2+y^2+2g_1x+2f_1y+c_1)+\lambda(x^2+y^2+2g_2x+2f_2y+c_2)=0.$$

The parameter of the system is λ.

4. The most convenient form of the equation to a coaxal system is $x^2+y^2+2\lambda x+c=0$. The two fixed points of the system are the intersections of the line $x=0$ and the circle $x^2+y^2+c=0$.

5. The point members of a coaxal system are called the "Limiting Points." There are two of them, and they may be real or imaginary.

EXAMPLES

1. Prove that the circles

$$x^2+y^2+4x+10y+8=0 \text{ and } x^2+y^2+6x+8y+44=0$$

cut orthogonally.

2. Prove that the circles

$$x^2+y^2+2x-5y=0 \text{ and } x^2+y^2-7x+4y-17=0$$

cut orthogonally.

3. (i.) If the circles

$$x^2+y^2-6x-14y+2=0 \text{ and } x^2+y^2-4x-10y+c=0$$

cut orthogonally, find c.

(ii.) Prove that if the circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \text{ and } x^2+y^2+2g_2x+2f_2y+c_2=0$$

cut orthogonally then their centres are conjugate points with respect to the circle

$$x^2+y^2=\tfrac{1}{2}(c_1+c_2).$$

4. A variable circle passes through the origin and cuts the circle $x^2+y^2-10x+12y-9=0$ orthogonally. Find the locus of its centre. (See Art. II. Ex. 2.)

5. A variable circle touches the x-axis and cuts the circle $x^2+y^2=9$ orthogonally. Find the locus of its centre. (See Art. II. Ex. 3.)

6. A variable circle touches the y-axis and cuts the circle $x^2+y^2-5x=0$ orthogonally. Find and draw the locus of its centre.

7. Find the radical axes of the following pairs of circles and draw the systems of (1) and (2) on squared paper.

$$(1)\begin{cases} x^2+y^2+6x+4=0. \\ x^2+y^2-10y+17=0. \end{cases}$$

$$(2)\begin{cases} x^2+y^2-9x+6y-5=0. \\ x^2+y^2-3x-6y+7=0. \end{cases}$$ (*N.B.*—Find a point on each of this pair.)

$$(3)\begin{cases} 4x^2+4y^2-12x+20y-15=0. \\ 3x^2+3y^2+6x-3y+8=0. \end{cases}$$

Verify in each case that the line of centres is perpendicular to the radical axis.

8. Write down the equation to a system of coaxal circles passing through the intersections of the circles

$$x^2+y^2-5x+4y-2=0$$
$$\text{and } x^2+y^2+3x-2y+6=0.$$

(i.) Find the member of the system that passes through the origin.

(ii.) Find the members of the system which touch XX'.

(iii.) To which member of the system is the tangent from the origin two units in length?

9. Write down the equation to the family of coaxal circles which pass through the intersections of the circles

$$x^2+y^2+6x-4y+8=0$$
$$\text{and } x^2+y^2-2x+6y-10=0.$$

(i.) Find the two points through which all the members pass and draw a few of the latter, showing the values of the parameter in each case.
(ii.) Find the values of the parameter correct to two decimal places, for these members which have a radius of 4 units.
(iii.) Find the equation to those members of the system which cuts the base circles orthogonally.

10. Write down the equation to the coaxal system which is defined by the intersection of the straight line $3x+4y=12$ and the circle $x^2+y^2-8x-4y+3=0$.
Which member of the system has its centre on XX' ?
Which member has its centre on the y-axis ?
Draw the given line and circle, and the two circles just found, showing in each case the value of the parameter.

11. A variable circle passes through the origin and has its centre on the line $x+y=3$. Prove that it passes through another fixed point and find it.
Steps : (i.) Write down the equation to a circle passing through O.
(ii.) State the condition that its centre lies on $x+y=3$, and use it to eliminate one of the co-ordinates of the centre in the equation to the circle.
(iii.) The result will be the equation to a coaxal system. Find the two points which define it.

12. A family of circles touches YY' at O. Write down their equation and find the member which passes through the point $(-1, 3)$.

13. In Example 1 it was proved that the circles

$$x^2+y^2+4x+10y+8=0 \quad\text{and}\quad x^2+y^2+6x+8y+44=0$$

cut orthogonally. Prove that the radical axis is the polar of the centre of one with respect to the other.

14. The variable circle $x^2+y^2-2x'x-2y'y+c=0$ cuts the fixed circle $x^2+y^2=9$ orthogonally.
(1) Prove that their radical axis is the polar of the centre of the variable circle with respect to the fixed one.
(2) If the radical axis always passes through the point (3, 6), find the locus of the variable centre.

15. On a given straight line OX a variable length OP is drawn. Q is a point such that the area of the square on OQ exceeds that of ΔOPQ by 4 sq. units. Prove that the locus of Q is a system of coaxal circles.

Draw some members of the system. What are the common points and what line is the radical axis ?

16. Which members of the coaxal system $x^2+y^2-2\lambda x-16=0$ have a radius of 5 units ?

17. Which of the following coaxal systems have real limiting points (Art. IX.) ?

$$\text{(i.)}\ x^2+y^2+2\lambda x+4=0.$$
$$\text{(ii.)}\ x^2+y^2-2\lambda x-6=0.$$
$$\text{(iii.)}\ x^2+y^2-2\lambda y+25=0.$$

Draw figures showing the limiting points when these are real.

18. *Circle of Apollonius.* A line AB is 6 units long. A variable point P moves so that the ratio $PA : PB$ is constant.

Prove that the locus of P is a member of a coaxal system. Draw the pencil of circles showing the limiting points.

19. If L is a limiting point of the coaxal system

$$x^2+y^2+2\lambda x+c^2=0,$$

prove that XX' cuts any member of the system in two points, P and Q, such that $OP \cdot OQ=OL^2$.

20. In last example verify that the tangents drawn from a point on the radical axis to all members of the system (including the limiting points) are equal.

21. Prove that the polar of a limiting point with respect to any member of the system is a fixed line passing through the other limiting point.

22. Q is a variable point on the radical axis of the pencil

$$x^2+y^2+2\lambda x+c^2=0.$$

A circle is described, having Q as centre and the tangent from Q to the given coaxal system as radius. Prove that this circle is a member of another coaxal system which passes through the limiting points of the first family and has XX' as radical axis. Prove also that the two systems cut orthogonally, and that one of them has real and the other unreal limiting points.

23. Prove that the polars of a given point with respect to a pencil of circles form a pencil of lines.

24. A variable circle passes through a fixed point and has its centre on a given line. Prove that it is one of a pencil of circles having the given point and another fixed point as vertices.

(*Hint.*—Take the fixed point as origin and a parallel to YY' as the line of centres.)

25. A circle is described with O as centre. A system of circles is drawn with their centres on XX' so as to cut the first circle orthogonally. Show that the system is a coaxal one.

26. Prove that the circle on the join of the limiting points of a coaxal system as diameter cuts all the family orthogonally.

27. If the circle $x^2+y^2+2gx+2fy+c=0$ cuts the circles

$$x^2+y^2+2ax+k^2=0$$
$$\text{and } x^2+y^2+2bx+k^2=0$$

orthogonally find g and c.

28. Tangents are drawn from a limiting point to the members of a coaxal system. Find the locus of the points of contact. Compare this result with Example 23.

(*Hint.*—State the condition that the tangent at (x', y') passes through L.)

29. Find the locus of the centres of all circles which cut orthogonally the circles

$$x^2+y^2+2x+3y-3=0$$
$$\text{and } x^2+y^2-8x-6y+21=0.$$

30. Prove that the circle $x^2+y^2=5$ cuts each of the circles $x^2+y^2+2x+2y+5=0$ and $x^2+y^2-4x-6y+5=0$ orthogonally.

Verify that the point (1, 2) lies on the first circle, and show that its polars with respect to each of the other two circles intersect on the first circle.

31. In last example (x_1, y_1) lies on the first circle.

(i.) Prove that its polars with respect to the other two circles intersect on the line $3x+4y+3x_1+4y_1=0$.

(ii.) Throw this equation into such a form as to show that its graph cuts the first circle at the point $(-x_1, -y_1)$.

(iii.) Verify that the point $(-x_1, -y_1)$ is the intersection of the polars, and hence prove that these intersect on the first circle.

(iv.) State the property which has been established by these steps.

32. A variable circle cuts a fixed circle orthogonally.

Find the locus of its centre, if the radical axis always passes through a given point.

33. Which members of the system $x^2+y^2+2gx-4=0$ touch the line $y=2x+3$?

Draw a graphical figure.

34. PQ and RS are equal lines of variable length, on OX and OY respectively.

A point T moves so that the square on OT is equal in area to the sum of the triangles PQT and RST. Prove that the locus of T is a pencil of circles.

35. A variable circle cuts a fixed circle orthogonally.

If the tangents from a given point to each circle are of equal length, prove that the locus of the centre of the variable circle is the polar of the point with respect to the fixed circle.

36. A variable circle cuts two fixed circles orthogonally; find the locus of its centre.

37. Prove that the polars of a point on the radical axis with respect to all members of a coaxal system intersect on that axis.

38. Prove that the three radical axes of three given circles are concurrent.

39. A system of circles touch a fixed line at a given point. Prove that the polars of all points on the fixed line pass through the given point.

40. Two circles cut each other orthogonally. Prove that the centre of one is the pole of the radical axis with respect to the other.

41. Two circles cut orthogonally. PQ is a diameter of one of them. Prove that Q lies on the polar of P with respect to the other.

42. Prove that the locus of the centres of all circles which pass through a fixed point and cut a given circle orthogonally is a straight line.

43. Find the condition that the two circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+2f_2y+c_2=0$$

cut orthogonally, in the following way:

(1) Let (x_1, y_1) be a point of intersection. Write down the equations to the tangents there to both circles, and state the condition that they be at right angles.

(2) State the conditions that (x_1, y_1) lie on each circle, and add the two results.

(3) Combine (1) and (2).

44. Prove Example 5, page 232, as follows:

(i.) Find the condition that the two circles cut orthogonally.

(ii.) Since P and Q harmonically separate A and B, they are therefore conjugate points with respect to the circle having O as centre. Work out the condition for this, taking $P\equiv(x_1, o)$ and $Q\equiv(x_2, o)$.

(iii.) Find the quadratic equation giving the points P and Q where XX' cuts the right-hand circle, and from it prove that the condition of (ii.) is true.

ANSWERS

CHAPTER I

(2.) 5. **(3.)** 3·45. **(4.)** 5·2. **(5.)** 4·12; ·97. **(6.)** 3·61; 5; 6·69; 4·78. **(7.)** 3·61; 7. **(8.)** 4; 3·76; 6·72. **(10.)** (2, 3); (3·6, 3·7); (−1, 1·5); (1·8, 1·9); (−2, 3); (−3·3, −1·5); (·85, 1·1); (0, 0). **(11.)** (2, 1); $\sqrt{5}$. **(12.)** (5, 0); (0, 2); (2·5, 1). **(13.)** (2, 0); (−2, 0); (0, 2); (0, −2); 4. **(14.)** (3, 0); (−3, 0); (0, 2); (0, −2); 12. **(15.)** (3, 0); (1, 0); (0, 3); 3. **(16.)** 15; 18·03. **(17.)** $\frac{\sqrt{a^2+b^2}}{2}$. **(18.)** $(-h, o)$. **(19.)** $\sqrt{29}$. **(20.)** 16. **(21.)** 21. **(22.)** 2. **(23.)** 1·2; point is (−1·2, 1·6). **(24.)** (2·79, 1·95). **(25.)** 7·21; 3·83. **(26.)** (3, 2). **(31.)** $xy=5$. **(32.)** $x^2+y^2=9$. **(33.)** $y^2=5x$. **(34.)** $x^2-y^2=4$. **(35.)** $C\equiv(4, 10)$; $D\equiv(10, 6)$; centre $\equiv(5, 5)$.

CHAPTER II

(2.) 4; 3; (4, 0) and (0, 3). **(3.)** $\frac{x}{5}+\frac{y}{3}=1$; $-\frac{x}{4}+\frac{y}{2}=1$; $-\frac{x}{6}-\frac{y}{9}=1$; $\frac{x}{x_1}-\frac{y}{y_1}=1$. **(4.)** 12. **(5.)** $\frac{x}{9/4}+\frac{y}{9/2}=1$; $-\frac{x}{3/5}+\frac{y}{1/2}=1$; $\frac{x}{5/2}-\frac{y}{5/3}=1$; $\frac{x}{2}-\frac{y}{4}=1$. **(6.)** (5, 0) and (0, 12); 13; 30; 4·62. **(7.)** 8; 5; 1·1. **(8.)** 1·6. **(10.)** 3 units along OX'. **(13.)** $x=3y$. **(14.)** $x+y=5$. **(15.)** $x-y=2$. **(16.)** $x+y=9$. **(17.)** (2·5, 1); 2·69; $2x=5y$. **(18.)** $\left(\frac{a}{2}, \frac{b}{2}\right)$; $\frac{a^2+b^2}{4}$; $\frac{x}{a}=\frac{y}{b}$. **(19.)** $N\equiv(x_1, o)$; $M\equiv(o, y_1)$; $Q\equiv\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$; $xy=5$. **(20.)** $3x=5y+4$; $4x+3y=1$; $8x-3y+18=0$; $5x+6y=0$. **(21.)** $4x+3y=24$; (0, 8). **(22.)** $\frac{x}{2a}+\frac{y}{b}=1$. **(23.)** $\frac{x}{a}+\frac{y}{b}=2$. **(24.)** (3, 2); (34, 20); (−5, −2·5). **(25.)** $x+5y=0$. **(26.)** 7·6 ft. **(27.)** $2y=5$. **(28.)** A straight line through O. **(29.)** A straight line perpendicular to AB. **(32.)** $\frac{1}{OA}+\frac{1}{OB}=\frac{1}{c}$. **(34.)** $\frac{x}{a}=\frac{y}{b}$; $QR=a-x'$ and $QS=b-y'$. **(35.)** $Q\equiv\left(\frac{h}{2}, \frac{k}{2}\right)$; $R\equiv\left(\frac{a+h}{2}, \frac{k}{2}\right)$. **(40.)** A straight line through O. **(45.)**

$P\equiv(0, -3)$; $Q\equiv(8, 0)$; $3x=8y+24$. **(47.)** Either $x+3y=12$ or $3x+y=12$. **(48.)** $\left(\frac{ab(a+b)}{a^2+ab+b^2}, \frac{a^2b}{a^2+ab+b^2}\right)$; $\left(\frac{ab^2}{a^2+ab+b^2}, \frac{ab(a+b)}{a^2+ab+b^2}\right)$. **(50.)** The bisector of $X\widehat{O}Y$. **(51.)** $\left(\frac{a}{3}, \frac{b}{3}\right)$. **(54.)** $4x+5y=13$. **(55.)** $x+5y=3$. **(56.)** $xy=x+2y$.

CHAPTER III

(1.) 47° 44′; 22° 23′; 2° 43′. **(2.)** 0°; 90°. **(3.)** (141° 20′; 123° ·42′; 17° 38′); (60° 15′; 39° 48′; 20° 27′). **(4.)** $\frac{x}{a}+\frac{y}{b}=1$; $\frac{x}{a}=\frac{y}{b}$; $\frac{2ab}{a^2-b^2}$. **(6.)** $x+2y=11$. **(7.)** $4x-5y=23$. **(8.)** $\frac{x}{a}+\frac{y}{b}=\frac{1}{2}$. **(9.)** $2x-3y=19$. **(10.)** $\frac{x}{a}+\frac{y}{b}=0$. **(12.)** $7x+5y=12$. **(13.)** $9x=7y$. **(14.)** $\frac{x}{b}=\frac{y}{a}$. **(16.)** $y=3x$; $x+y=4$; (1, 3). **(17.)** $3x+4y=4$. **(20.)** $\frac{1}{3}$; $-·48$; $\frac{5}{7}$; $-1·9$; $·8$; $·8$; $-\frac{b}{a}$. **(21.)** 39° 17′; 123° 41′ 30″; 118° 11′. **(22.)** $\frac{5}{2}$; $-\frac{4}{7}$. **(23.)** $\frac{7}{2}$; $-\frac{2}{5}$. **(30.)** 72° 1′. **(31.)** $9\frac{7}{16}$. **(34.)** (2, 4). **(35.)** $\left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$. **(36.)** $y=\frac{2}{3}x+4$; $y=-\frac{5}{8}x+3$; $y=-·8x-2$; $y=1·7x$. **(37.)** $\left(-\frac{b}{2m}, \frac{b}{2}\right)$. **(38.)** A line through the origin. **(39.)** (4, 2). **(40.)** $\left(\frac{-2bm}{1+m^2}, \frac{b(1-m^2)}{1+m^2}\right)$. **(41.)** $x+y=a$; $x+y=b$; $D\equiv\left(\frac{a}{2}, \frac{a}{2}\right)$; $E\equiv\left(\frac{b}{2}, \frac{b}{2}\right)$. **(42.)** $-\frac{a}{b}$; $\left(\frac{b^2}{a}, 0\right)$. **(43.)** $\frac{a^2+ab+b^2}{(a+b)^2}$. **(46.)** $x-6y+12=0$; $x-6y+4=0$. **(47.)** (4, 7·5). **(50.)** $\frac{6ab}{9a^2-b^2}$. **(52.)** $\frac{a+2b}{a-2b}$. **(53.)** 69° 38′. **(54.)** $\frac{m^2-1}{2m}$; 67° 30′ or 157° 30′. **(55.)** $y=mx+b$; $P\equiv\left(-\frac{2b}{m}, -b\right)$; $S\equiv\left(a, \frac{ma}{2}\right)$. **(56.)** A straight line through C.

CHAPTER IV

(1.) On opposite sides. **(2.)** The points (1, 0) and (3, −5) are on the origin side. **(3.)** 3·2; 6·13; 4·47; 5·81; 9·06. **(4.)** 3·59. **(5.)** 12·8. **(7.)** 7·06. **(8.)** $\frac{ab(a^2-b^2)}{a^2+b^2}$. **(9.)** $4x+3y=20$. **(10.)** $x+y-6=\pm5\sqrt{2}$. **(11.)** 12. **(12.)** 8. **(13.)** 9. **(14.)** $\frac{2x+5y+3}{\sqrt{29}}=\pm\frac{3x-y+4}{\sqrt{10}}$; $y=3$ and $x=-\frac{2}{3}$; $\frac{x-y}{\sqrt{2}}=\pm\frac{3x+4y}{5}$. **(15.)** $2x+y=3$; $x+3y=4$. **(16.)** 68° 12′. **(17.)** 303° 41′. **(18.)** $x\cos 50°+y\sin 50°=2$.

CHAPTER V

(1.) $x^2+y^2=1$; $x^2+y^2=9$; $x^2+y^2=49$. **(2.)** 4 units. **(4.)** 2·10. **(5.)** Circle on AB as diameter. **(6.)** Circle with centre O and radius of 5 units. **(7.)** $(-1\cdot8, -2\cdot6)$ and $(1, 3)$. **(10.)** $(-8, 1\cdot6)$. **(11.)** $(\frac{18}{25}, \frac{26}{25})$. **(13.)** ±5. **(14.)** 52° 14′ or 127° 46′. **(16.)** Circle on AB as diameter ; radius 5 units. **(18.)** 1·90. **(19.)** $\dfrac{b}{\sqrt{1+m^2}}$; $\dfrac{n}{\sqrt{l^2+m^2}}$. **(21.)** and **(22.)** Circles with centre O. **(24.)** (3 cos 48°, 3 sin 48°). **(25.)** (5 cos 53° 8′, 5 sin 53° 8′). **(26.)** (17 cos 151° 56′, 17 sin 151° 56′). **(28.)** A circle with centre O and radius $\dfrac{a}{2}$. **(30.)** The line $x=y$. **(31.)** A line through O of gradient $-\frac{5}{3}$. **(34.)** The circle $x^2+y^2=10$. **(35.)** 38° 59′ ; 79° 6′. **(38.)** The diameter through B. **(39.)** $x^2+y^2=17$. **(40.)** $x^2+y^2=x_1^2+y_1^2$. **(41.)** (1, 1), (2, 3), and (−3, 1) are inside the circle. **(42.)** 4·47 ; 6·56 ; 5·74 ; 7.

CHAPTER VI

(1.) $5x+2y=29$. **(2.)** $3x+4y=25$; $x-3y+20=0$; $4x+5y+41=0$; $x-7y=50$. **(3.)** $16\frac{2}{7}$. **(4.)** $\frac{1}{2}$; -2. **(5.)** $2x=9y$. **(6.)** $-\frac{2}{3}$. **(7.)** $8x=3y$. **(8.)** $5x=2y$. **(9.)** $\frac{1}{4}$. **(11.)** $x^2+y^2=12$. **(13.)** $5x+7y=2$; $8x-3y=4$; $4x+6y+5=0$. **(15.)** $(\frac{25}{7}, -\frac{5}{7})$ **(16.)** $(6, -18)$. **(17.)** $(1\cdot41, 1\cdot01)$; $(-\cdot68, -1\cdot59)$. **(18.)** $(2, -2)$; $(-\frac{14}{17}, \frac{48}{17})$. **(22.)** $\left(\dfrac{a^2}{5}, -\dfrac{2a^2}{5}\right)$.

CHAPTER VII

(1.) $(2\cdot8, 3\cdot6)$. **(2.)** $(2\cdot5, 4\cdot5)$. **(3.)** $(\cdot5, -1\cdot5)$. **(4.)** $(2\cdot2, 3\cdot6)$; $(4\cdot33, 4\cdot67)$. **(5.)** Externally as 2 : 1. **(6.)** Internally as 17 : 10. **(7.)** Externally as 31 : 50. **(9.)** $(5, -5)$. **(11.)** $\left(-\dfrac{ab}{a-b}, \dfrac{ab}{a-b}\right)$. **(13.)** 2 and $-\frac{1}{2}$. **(14.)** $-\frac{14}{5}$ and $\frac{1}{56}$ (approx.). **(15.)** 0 and $\frac{23}{2}$. **(21.)** 5+2 cos 32° ; 5+2 sin 32°. **(22.)** 3+3 cos 142° ; 4+3 cos 142°. **(24.)** 18. **(25.)** $\left(\dfrac{x_1+x_2+x_3}{3}; \dfrac{y_1+y_2+y_3}{3}\right)$. **(26.)** $(\frac{14}{3}, 2)$. **(27.)** $\left(\dfrac{a}{3}, \dfrac{b}{3}\right)$. **(28.)** $5x+2y=0$. **(29.)** $5x+4y=0$. **(30.)** $y+2x=0$.

CHAPTER VIII

(1.) $y-1=\frac{3}{2}(x-4)$; $y-2=-\frac{4}{5}(x+3)$; $y+2\cdot8=\frac{4}{3}(x+3\cdot3)$; $y+1\cdot4=-\frac{5}{6}(x-2\cdot5)$. **(2.)** $5x-3y=12$. **(3.)** $\left(\dfrac{h^2+k^2}{h}, \dfrac{h^2+k^2}{k}\right)$. **(4.)** $5x+3y=17$. **(5.)** $y=x+1$; $2y=x+8$; $x-6y=7$; $\dfrac{x}{h}=\dfrac{y-b}{k-b}$. **(6.)**

$(x+3=0;\ m=\infty)$; $(y-1=0;\ m=0)$. **(7.)** $-\frac{1}{6}$; $-\frac{11}{41}$; $-\frac{9}{27}$; $-\frac{13}{46}$. **(8.)** $(x-4y=0;\ \lambda=-\frac{3}{16})$; $(4x+y=17;\ \lambda=\frac{22}{13})$; $(x-5y+1=0;\ \lambda=-\frac{5}{16})$; $(x=2y+2;\ \lambda=\frac{1}{16})$; $(x=4;\ \lambda=\frac{5}{4})$; $(x+y=5$ and $x+16y=20;\ \lambda=7$ and $-\frac{27}{44})$; vertex of pencil (4, 1). **(9.)** $2x+5y=20$; $5x-2y=21$; quad. $=14{\cdot}2$. **(18.)** $5\alpha+2\beta=1$.

CHAPTER IX

(2.) 8. **(3.)** $5x-2y=0$ and $x+3y=0$; $5x+4y=0$ and $3x+2y=0$; $x=0$ and $6x-5y=0$. **(6.)** ± 20. **(7.)** $b=-8$. **(8.)** $6m^2-4m-3=0$; $3m^2-5m+2=0$; $m^2+7m-4=0$; $48^\circ\ 7'$ and $155^\circ\ 51'$. **(9.)** 81° (about); $29^\circ\ 44'\ 30''$. **(10.)** $58^\circ\ 41'$. **(11.)** 0°; 90°. **(12.)** $(\frac{8}{9}, 2)$. **(13.)** $(-\frac{3}{13}, -\frac{7}{13})$. **(14.)** $80x^2+108xy+35y^2=0$. **(15.)** $6^\circ\ 9'$. **(16.)** $28^\circ\ 4'$. **(17.)** $l^2+m^2=\frac{1}{8}$. **(18.)** $5l^2+5m^2=2$. **(19.)** $x^2+y^2=40$

CHAPTER X

(1.) $x^2+y^2-10x-6y+18=0$. **(2.)** $x^2+y^2-8x-4y+11=0$; $x^2+y^2+12x-10y+57=0$; $x^2+y^2-6y-16=0$; $x^2+y^2+4x-12=0$. **(3).** $C\equiv(1,\ 3)$, $r=2$; $C\equiv(-2,\ 4)$, $r=3$; $C\equiv(3,\ -4)$, $r=5$; $C\equiv(\frac{4}{3}, -\frac{5}{3})\cdot r=\sqrt{\frac{127}{18}}$. **(4.)** $128{\cdot}8$; $47{\cdot}12$. **(5.)** Either $(x+5)^2+(y-2)^2=r^2$ or $x^2+y^2+10x-4y+c=0$; point member $\equiv x^2+y^2+10x-4y+29=0$. **(6.)** $x^2+y^2-6x+2y+10=0$. **(7.)** $x^2+y^2+6x-8y=0$. **(8.)** A circle with the given point as centre. **(9.)** $x^2+y^2-8x\pm 6y=0$. **(10.)** $x^2+y^2-4x-6y+3=0$. **(11.)** $x^2+y^2-3y=0$. **(14.)** $x^2+y^2-2cy-a^2=0$. **(16.)** $x^2+y^2-30y+189=0$. **(17.)** (1, 1) and (2, 3). **(18.)** $(-3,\ 2)$; $(1, 5)$; 5; $(-1, \frac{7}{2})$. **(19.)** (5, 2). **(20.)** $(-3, 4)$. **(22.)** c. **(23.)** $y=x+2$; (3, 5) and $(-1,\ 1)$. **(24.)** (2, 2) and (6, 8); $3x-2y=2$. **(25.)** (3, 3). **(26.)** $4x+3y+9=0$. **(27.)** $(x+y=4;\ 9\sqrt{2})$; $(2x+3y=23;\ 3\sqrt{13})$; $\left(x-3y+6=0;\ \dfrac{7\sqrt{10}}{3}\right)$; $(3x-4y=12;\ 10)$. **(28.)** $2x^2+2y^2-6x-7y=0$. **(29.)** $38{\cdot}6$; $3x=19y$. **(31.)** $\dfrac{gl+fm+1}{\sqrt{l^2+m^2}}$; a pair of parallel straight lines. **(32.)** $c=0$; $c=g=0$; the perpendicular to the given line at the point. **(37.)** $x^2+y^2-2by=0$.

CHAPTER XI

(1.) $-\frac{1}{2}$. **(3.)** $-\frac{3}{11}$; $-\frac{7}{6}$; $-\frac{1}{2}$. **(4.)** -2. **(5.)** $(2{\cdot}77,\ -6{\cdot}07)$; $(1{\cdot}23,\ {\cdot}07)$. **(6.)** $4x+7y=40$. **(7.)** $x-3y+13=0$; $3x+11y=63$; $x-2y+12=0$; $gx+fy=0$. **(8.)** $3x+y=11$; $11x-3y+29=0$; $2x+y=1$; $fx-gy=0$. **(9.)** $35{\cdot}27$. **(10.)** $14x+13y+48=0$; $8x-19y=40$; $19x-6y+57=0$; $15x+30y=32$; $gx+fy+c=0$. **(12.)** $5x-8y=12$. **(13.)** $\sqrt{71}$; $\sqrt{85}$; $\sqrt{82}$; $\sqrt{3{\cdot}2}$; $\sqrt{c}$. **(16.)** Without; within; without. **(17.)** The origin is without the first and within the second. **(18.)** $x^2+y^2-2ax-2ay+a^2=0$. Put $a=10$ and 2. **(19.)** $5x+3y+30=0$; $y+f=m(x+g)$ where $-\dfrac{1}{m}=$ gradient of parallel system. **(25.)** $x=2y$; $2x+y=5$.

CHAPTER XII

(3.) $x^2+y^2=23$. **(4.)** $x^2+y^2=30$. **(5.)** $(4\cdot5,\ 2\cdot5)$. **(6.)** $3x+4y=12$. **(7.)** $2x+4y=7$; $3x-5y+9=0$; $y+3=0$; $9x+3y+8=0$. **(8.)** 5. **(10.)** $(4,\ -6)$. **(11.)** ([illegible], -3). **(14.)** $(lr^2,\ mr^2)$. **(20.)** $21x+29y=12$. **(21.)** $6x+7y+25=0$; $3y-3x=55$; $5x-3y+18=0$; $8x-11y+6=0$; $27x+8y+43=0$. **(23.)** $2x-3y+5=0$. **(25.)** $x^2+y^2-6x+2y=0$. **(26.)** $x^2+y^2-4x+8y=0$. **(27.)** $y=x+14$. **(30.)** 4. **(31.)** $x+y=ka^2$; $a^2(x+y)=kxy$; $kxy=a^4$. **(33.)** $x^2+y^2+6x-8y+4=0$. **(34.)** $gx+fy+c+x'(x+y)=0$. **(36.)** $x^2+y^2+\dfrac{2a^2}{y'}y-a^2=0$. **(37.)** $x^2+y^2-8x-12y+3=0$. **(38.)** $x'+y'$ where given point $\equiv (x',\ y')$. **(39.)** $8x-y=14$. **(40.)** $x^2+y^2-9x-7y+17=0$. **(41.)** $x+y+23=0$. **(46.)** (i.) The line of centres; (ii.) a circle.

CHAPTER XIII

(3.) 80. **(4.)** $10x-12y+9=0$. **(5.)** The straight lines $x=\pm3$. **(6.)** $y^2=5x$. **(7.)** $6x+10y=13$; $2y=x+2$; $72y=60x+77$. **(8.)** $2x^2+2y^2-6x+5y=0$; $4x^2+4y^2+24x-17y+36=0$; $4x^2+4y^2-8x+7y+4=0$; $2x^2+2y^2+2x-y+8=0$. **(9.)** $(1,\ 1)$ and $(-\frac{101}{41},\ -\frac{7}{41})$; $12\cdot23$ or $-\cdot23$; $13x^2+13y^2+38x-2y+14=0$ and $7x^2+7y^2+26x-8y+20=0$. **(10.)** $(x^2+y^2-5x-9=0;\ \lambda=1)$; $(3x^2+3y^2+20y-87=0;\ \lambda=\frac{8}{3})$. **(11.)** $(3,\ 3)$. **(12.)** $x^2+y^2+10x=0$. **(14.)** $x+2y=3$. **(15.)** Take O as origin and OX as axis of x. Fixed points are $(2,\ 0)$ and $(-2,\ 0)$. Radical axis is OY. **(16.)** $x^2+y^2\pm6x-16=0$. **(17.)** Nos. (i.) and (iii.). They are $(2,\ 0)$ and $(-2,\ 0)$; $(5,\ 0)$ and $(-5,\ 0)$. **(27.)** $c=-k^2$; $g=0$. **(29.)** $10x+9y=24$. **(32.)** The polar of the point *w.r.* fixed circle. **(33.)** $x^2+y^2-2x-4=0$; $x^2+y^2-22x-4=0$. **(36.)** The radical axis of the two circles.

THE END

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